• **An Example of a Typical Related Rates Problem.** The area of a triangle is increasing at the rate 4 cm^2/min. If the height is twice the base how much is the base increasing when its length is 6 cm? To solve this problem we need to to know that the area A of a triangle with base b and height h is given by $A = \frac{1}{2}bh.$, The parameters $A = A(t), b = b(t)$ and $h = h(t)$ are all functions of time, t. Since $h(t) = 2b(t)$, for all t the equation for the area can be reduced to $A(t) = (b(t))^2$. This is referred to as the *constraint equation.* Using the the chain rule to differentiate both sides of the constraint equation with respect to t, we find $A'(t) = 2b(t)b'(t)$. Evaluating at that particular time, t_0, when the base $b(t_0)$ is 6 cm, and the rate of change $A'(t_0)$ of the area is $A'(t_0)$ is 4 cm^2/min., we solve for $b'(t_0)$ to find that $b(t)$ is increasing at the rate of $\frac{1}{3}$ cm/min.

• **The General Related Rates Poblem.** In general, two (or more) variables which depend on time, t, are given. The related rate problem asks for the rate of change (i.e. the derivative) of one of the dependent variables at a specific time t_0. We are either given or can compute the rate of change and values of the other variables at t_0. The problem is always constrained by a relation involving the dependent variables say, $x(t)$ and $y(t)$, of the form

$$F\big(x(t), y(t)\big) = C.$$

In the example given above, the dependent variables can be reduced to $A = A(t)$ and $b = b(t)$ and the constraint equation can be taken as $A(t) - b^2(t) = 0$ or equivalently as $A(t) = b^2(t)$.

• **Method of Solution.** Implicitly differentiating both sides of the constraint equation with respect to t with the help of the chain rule, produces another equation of the form

$$G\big(x(t_0), y(t_0), x'(t_0), y'(t_0)\big) = 0,$$

whose only unknown is either $x'(t_0)$ or $y'(t_0)$. We call this equation the *derived equation* and solve it for the only unknown present. The independent variable t, as well as the specified time t_0 are usually suppressed (i.e. assumed but not explicitly written). In the above example, the derived equation can be succinctly written as $A' - 2bb' = 0$ or equvalently $A' = 2bb'$. Here A' and b were known to be 4 and 6 respectively, so we simply solved for b'; the only unknown.

• **Example:** A 20 foot ladder is leaning against a building. Suppose the ladder is slipping so that, at a particular time t_0, its top is moving down the wall at the rate of $\frac{3}{2}$ feet per second. At what rate is the bottom of the ladder moving away from the wall when it is 16 feet from the wall?

Letting $y(t)$ be the height of the ladder and $x(t)$ be the distance that the base of the ladder is from the wall at time t, The Pythagorean theorem gives the constraint equation $x^2(t) + y^2(t) = 20^2$. Differentiating both sides with respect to t, gives $2xx' + 2yy' = 0$. We know $x(t_0) = 16$. We can compute $y(t_0)$ from the constraint equation, and since $y'(t_0) = \frac{3}{2}$, we need only solve the last equation $2xx' + 2yy' = 0$ for the unknown $x'(t_0)$.

Problem 1. Finish the solution to the above ladder problem. (Answer $1\frac{1}{8}$ ft./sec.)

Problem 2. Ann, who is 5 feet tall is walking away from a 9 foot lamp-post at the rate of 2 feet/second. How fast is the tip of her shadow moving when she is 25 feet from the base of the lamp-post? (Answer $\frac{9}{2}$ ft./sec.)