• **Chain Rule.** If $y = f(u(x))$ is differentiable at $u(x)$ and $u(x)$ is differentiable at $x$, then the composition $(f \circ u)(x) = f(u(x))$ is differentiable at $x$ and its derivative is given by:

$$D_x f(u(x)) = \frac{dy}{du}\frac{du}{dx},$$

with the understanding that $\frac{dy}{du}$ and $\frac{du}{dx}$ are evaluated at $u(x)$ and $x$ respectively.

**Remarks.** (i) As a memory aid think of cancelling the $du$'s.

(ii) This rule is actually a rule for differentiating composite functions and as such might be more aptly renamed.

(iii) In practice, the hard part is to write the function $f(x)$ to be differentiated as a composite function $f(u(x))$. More often than not $u(x)$ is that part of the function in parenthesis. For example if $f(x) = (x^2+1)^{13}$, set $u = (x^2+1)$.

• **Power Rule.** Setting $u(x) = g(x)$, one finds

$$f(x) = (g(x))^n \Longrightarrow f'(x) = ng'(x)(g(x))^{n-1},$$

where one must add the usual disclaimer $g(x) \neq 0$ if $n < 1$. Setting $n = 1/2$ and $-1$ respectively one gets the frequently used differentiation formulas:

$$D_x\sqrt{g(x)} = \frac{g'(x)}{2\sqrt{g(x)}} \text{ and } D_x\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{(g(x))^2};$$

both of which are worth memorizing.

• **Extended Chain Rule.** The above chain rule can be extended to a function $f(u(v(x)))$ which is the composition of a function $u(v(x))$ which in turn is a composition of a function $v(x)$ producing a differentiation formula of the form:

$$D_x f(u(v(x))) = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

*Problem.* Use the chain rule to prove the extended chain rule. What differentiability conditions are needed?

*Problem.* Use the extended chain rule to differentiate

$$f(x) = \sqrt{(x^2+1)^3 + 2(x^2+1)^2 + 1}.$$

*Hint: Set* $f(u) = \sqrt{u}, u(v) = v^3 + 2v^2 + 1$ *and* $v(x) = x^2 + 1$. *Don't leave any $u$'s or $v$'s in your answer; only $x$'s.*

*Problem.* Let $g(x)$ be a differentiable function which is not zero at $x$. Derive the formula $D_x\frac{1}{g^n(x)} = -n\frac{g'(x)}{g^{n+1}(x)}$ by using the chain rule and the power rule for differentiation.

*Problem.* Use the chain rule to differentiate $|x|$ by writing $|x| = \sqrt{x^2}$.

*Problem.* Let $f(u) = u^3$ and $u(x) = x^2 + 1$ Differentiate the composition $(f \circ g)(x)$ both with and without the chain rule. Compare your answers.