Semigroup problem for \( \text{Aff}^+ \) and related groups.

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Introduction of Problem

For certain solvable groups, we wish to prove that:

1. the closure of the semigroup generated by subsets not contained in any maximal semigroup with non-empty interior is a group.

2. arbitrarily close to any such subset is one that generates a dense semigroup.
Background

This problem have been solved for certain groups:

- Compact Extension of Nilpotent groups
- $\mathbb{R}^n$

Why is the problem interesting?

This problem is an obstruction to transitivity of extensions of hyperbolic systems.
Definitions

- A subset $S$ of a group $G$ is called *good* if the closure of the semigroup it generates is a group not contained in any connected co-dimension 1 subgroup.
- A subset $S$ of a group $G$ is called *great* if it generates a dense semigroup.
Our group

Define

\[ G_n := \left\{ \begin{pmatrix} a & b \\ 0 & l_n \end{pmatrix} : a \in \mathbb{R}^+, b \in \mathbb{R}^n \right\} \cong \mathbb{R} \times \phi \mathbb{R}^n. \]

Note that \( G_1 = \text{Aff}^+ \) and is the simplest solvable Lie group that is not nilpotent. It is in fact the unique simply connected non-abelian 2-dimensional Lie group.
Our group

We consider a further generalization of Aff$^+$ given by

$$H_{mn} := \mathbb{R}^m \rtimes \phi \mathbb{R}^n \cong \mathbb{R}^{m-1} \times G_n$$

where $\phi$ is central. After identifying the $a$ coordinate in $G_n$ by its natural log, multiplication is given by

$$(v, a, b)(v', a', b') = (v + v', a + a', b + e^a b')$$

for $v, v' \in \mathbb{R}^{m-1}, a, a' \in \mathbb{R}, b, b' \in \mathbb{R}^n$.

Some notable facts

- Solvable
- Exponential
- Simply Connected
Definitions

- A maximal semigroup with non-empty interior of a topological group $G$, is a proper subsemigroup $M$ of $G$ with non-empty interior such that $M$ is not a group and the only subsemigroups of $G$ containing $M$ are $G$ and $M$.
- A subset $S$ of a topological group $G$ is called separated if it is not contained in any maximal semigroup with non-empty interior.
Lawson’s Theorem

Theorem

The maximal subsemigroups $M$ with non-empty interior of a simply connected Lie group $G$ with $G/\text{Rad } G$ compact are in one-to-one correspondence with their tangent objects

$$L(M) = \{x \in L(G) : \exp tx \in M \text{ for } t \geq 0\}$$

and which are precisely the closed half-spaces with boundary a subalgebra. We will call the images of the hyperplane subalgebras under the exponential map border subgroups.
Exponential map

Lemma

The exponential map $\gamma_{mn} \to H_{mn}$ is an analytic bijection with analytic inverse.

Proof.

If $m = (a, b)$ then

\[ \exp(m) = \begin{cases} (a, \frac{b}{a}(e^a - 1)) & \text{if } a \neq 0 \\ (0, b) & \text{if } a = 0, \end{cases} \]

\[ \log(x, y) = \begin{cases} (x, y \frac{x}{e^x - 1}) & \text{if } x \neq 0 \\ (0, y) & \text{if } x = 0. \end{cases} \]
Maximal semigroups of $G_1$

Every co-dimension 1 subspace of $g_1$ is a subalgebra, as it is of dimension 1. The border subgroups are the exponential images of these subalgebras. If the subalgebra is $\{t(a, b) : t \in \mathbb{R}\}$ then the corresponding border subgroup is

$$\left\{ \left( ta, \frac{b}{a} e^{ta} - 1 \right) : t \in \mathbb{R} \right\}$$

which is the curve

$$y = l(e^x - 1)$$

with $l = \pm \infty$ corresponding to the curve $x = 0$. 
Facts about border subgroups

- Each nonzero point in $\mathbb{R}^2$ belongs to a unique border subgroup.
- For positive values of $l$, the curve is contained in the first and third quadrants. For negative values of $l$, the curve is contained in the second and fourth quadrants.
- Fix some $l \leq 0$. Then, for points $z$ in the fourth quadrant, if $l_z$ is the slope of the curve corresponding to $z$, then $l_z < l \iff z$ is below the curve corresponding to $l$. For points $z$ in the second quadrant, $z$ is below the curve corresponding to $l \iff l_z > l$. 
Types of automorphisms

We have the following Lie group automorphisms of $H_{mn} := \mathbb{R}^m \times_{\phi} \mathbb{R}^n \cong \mathbb{R}^{m-1} \times G_n$:

(A) Any automorphism $\psi$ of $\mathbb{R}^{m-1}$ extends to an automorphism $\psi \times \text{id}$.

(B) Any automorphism $\psi$ of $\mathbb{R}^n$ extends to an automorphism which is the identity on the $a$ coordinate of $G_n$ and on each coordinate of $\mathbb{R}^{m-1}$.

(C) For a fixed $(v, a, b) \in H_{mn}$, there exists an automorphism defined by

$$(x, y, z) \mapsto \left(x - \frac{y}{a} v, y, z - b \frac{e^{y} - 1}{e^{a} - 1}\right)$$
Short Exact Sequence Lemma

Let the following

\[ 0 \to B \to G \to A \to 0 \]

be an exact sequence of topological groups. Let \( S \subseteq G \) and let \( U \) be the semigroup generated by \( S \). Let \( U(X) \) in this lemma denote the semigroup generated by a subset \( X \). Then:

1. \( \pi(U) = \pi(U) \).
2. \( \overline{U} \) is a group, then so is \( U(\pi(S)) \).
3. \( S \) is great so is \( \pi(S) \).
4. \( U(\pi(S)) \) is a group, \( \pi(U) \) is closed and \( \overline{U} \cap B \) is a group, then \( \overline{U} \) is a group.
5. If \( \pi(S) \) is great, \( \pi(U) \) is closed and \( \overline{U} \cap B \) is great, then so is \( S \).
6. \( S \) is separated so is \( \pi(S) \).
7. \( B = \mathbb{R}^n \) and \( G = \mathbb{R}^n \rtimes A \), with \( A \) second countable, and suppose \( \overline{U} \cap B \) is good. Then, \( \pi(U) \) is closed.
Good subsets of $G_1$

Lemma

Let $S \subseteq H_{mn}$ be separated. Let $U$ be the semigroup generated by $S$ and suppose $\overline{U}$ contains $z_0 = (w, a, b), z = (w', a', b')$ with $a < 0, a' > 0$. Then, $\overline{U}$ contains $\left(0, 0, \frac{b}{1-e^a} + \frac{b'}{e^{a'}-1}\right)$.
Good subsets of $G_1$

Lemma
For any separated subset $S$ of $G_1$, if $U$ is the semigroup generated by $S$, $U$ contains up to automorphism elements $(a, 0), (c, d)$, $a, d > 0, c < 0$. In fact, up to an arbitrary small perturbation, $S$ contains such elements.
Good subsets of $G_1$ 

Lemma 

Suppose a semigroup $U$ contains elements $z, z'$ respectively in the interior the second and fourth quadrant, such that $l_z > l_{z'}$. Then, $U$ contains an element in the interior of the third quadrant.

$z = (\ln a, b)$

$z' = (\ln a', b')$
Good subsets of $G_1$

Lemma

Suppose $S \subseteq G_1$ is separated with generated semigroup $U$ containing $(a, 0), (a', b')$, $a, b' > 0, a' < 0$. Then, $U$ contains $(c, d), c, d < 0$.

- Suppose $S$ is separated and contains $z, z_0$, respectively. Since $y \geq 0$ is a maximal semigroup of $G_1$, $S$ contains an element $z' = (c, d)$ with $d < 0$.
- If $c < 0$ we are done.
- If $c = 0$, then

$$z_0^k z' = \left(ka', e^{la'}d + b' \frac{1 - e^{ka'}}{1 - e^{a'}}\right).$$

As $k \to \infty$, $e^{ka'} \to 0$, giving the desired result.
**Good subsets of $G_1$**

So we may assume $c > 0$. We proceed via contradiction. Suppose $U$ contains no points in the interior of the third quadrant. Define

\[ L := \{ l_z : z \in \text{Interior of Second Quadrant} \} \]

\[ L' := \{ l_z : z \in \text{Interior of Fourth Quadrant} \}. \]

By previous lemma, we have $\sup L \leq \inf L' = l$. 
Good subsets of $G_1$

Lemma

Let $S \subseteq \mathbb{R}^n$. Let $X$ be an arbitrary one dimensional subspace of $\mathbb{R}^n$ and suppose $\tau : \mathbb{R}^n \rightarrow X$ is any projection. Then, if $\tau(S)$ is separated for each such $X, \tau$, then $S$ is separated.
Good subsets of $G_1$

Theorem

Suppose $S \subseteq H_{mn}$ is separated. Let $U$ be the generated semigroup. Then, $\overline{U}$ is a group.
Good subsets of $G_1$

- Note that

$$0 \rightarrow B = \mathbb{R}^n \rightarrow H_{mn} \rightarrow C \times A = \mathbb{R}^m \rightarrow 0.$$  

So by the Exact Sequence Lemma, it suffices to show that $\overline{U} \cap B$ is good, for which we will use the previous lemma. Call $\overline{U} \cap B = S'$. We may assume $\overline{U}$ contains $(0, a, 0)$ for $a > 0$.

- Choose an arbitrary one dimensional subspace $X$ of $B$ and projection $\tau : B \rightarrow X$. We may use an automorphism to let $X$ be the $B_1$ coordinate and let $B_2, \ldots, B_n$ be the kernel of $\tau$. We will show that $\tau(S')$ is separated.
Good subsets of $G_1$

- Let $\pi$ be projection onto the $A$ and $B_1$ coordinate which is isomorphic to $G_1$.

- By previous lemmas, we may assume $U$ contain elements $z' = (x'', a', b')$, $z'' = (x'', a'', b'')$ with $a' < 0$, $b' > 0$, $a''$, $b''_1 < 0$.

- Thus, using a previous lemma, we see that $S'$ contains $(0, 0, \frac{b'}{1-ea'})$ and $(0, 0, \frac{b''}{1-ea''})$. Thus, $\tau(S')$ contains positive and negative elements and is hence separated.
Semigroup Problem 2

For certain groups, we wish to show that arbitrarily close to any good subset is a great subset.
Great subsets of $\mathbb{R}^n$

**Lemma**

Let $S \subseteq \mathbb{R}^n$. If $e_1, e_2, \ldots, e_n \in S$ and

$$v := (v_1, \ldots, v_n) \in \overline{U(S)}$$

with $v_i < 0$, $\{1, v_1, \ldots, v_n\}$ $\mathbb{Z}$-linearly independent then $S$ is great.

This follows from Kronecker's Theorem on Diophantine Approximations:

**Theorem**

If $v$ is defined as above then $\mathbb{Z}v \mod 1$ is dense in $[0, 1]^n$. 
Second Semigroup Conjecture for $\mathbb{R}^n$

Lemma

For any $k$, the set of great $k$-tuples of $\mathbb{R}^n$ is dense in the set of separated $k$-tuples.

Proof.

- If $S$ is a good subset then after applying an automorphism, $S$ contains $e_1, \ldots, e_n$ and

$$\mathbf{v} := (v_1, \ldots, v_n) \in \overline{U(S)}$$

with $v_1, \ldots, v_n < 0$.

- For any nonzero $\alpha \in \mathbb{R}$, the set of $\mathbf{v} \in \mathbb{R}^n$ with $\{\alpha, v_1, \ldots, v_n\}$ $\mathbb{Z}$-linearly dependent has Lebesgue measure 0 and its complement is full.

- So given a good $S$, we can perturb some elements to make $\{1, v_1, \ldots, v_n\}$ $\mathbb{Z}$ linearly independent.
Lemma

Let $S \subseteq H_{mn}$ be good. Then, there exists a Lie group automorphism $\Phi$ of $H_{mn}$, such that

$$(x, \ln a, 0), (x_i, \ln c_i, |1 - c_i|e_i) \in \Phi(S)$$

where $\ln a > 0$, $e_i$ is the $i$th standard basis vector for $B \cong \mathbb{R}^n$.

Additionally, we can choose $\Phi$ such that either $\Phi(S)$ contains $(x', \ln a', 0)$ for $a' < 1$ or $c_1 < 1$. 
Second Semigroup Conjecture for $H_{mn}$

**Theorem**

For any $l$, the set of great $l$-tuples of $H_{mn}$ is dense in the set of separated $l$-tuples of $H_{mn}$. 
Second Semigroup Conjecture for $H_{mn}$

Let $S$ be separated and $U$ be the semigroup generated by $S$. Suppose $\pi_1$ is projection onto $\mathbb{R}^n$. Then by arbitrarily small perturbation, we may assume $\pi_1(S)$ is great and $S$ contains $z = (w, \ln a, 0), \ln a > 0, z_i = (w_i, \ln c_i, |1 - c_i|e_i)$ with either $c_1 < 1$ or an element $z' = (w', \ln a', 0)$ with $a' < 1$.

Additionally, $U$ contains element $z'' = (w'', \ln c, b)$ with each $b_i < 0, \ln c < 0$ and $b_{i+1} < b_i - 2(1 - c)$.

However, $S$ may no longer be separated.
Second Semigroup Conjecture for $H_{mn}$

- By the Exact Sequence Lemma, it suffices to show that $\overline{U} \cap B$ is great.
- By a previous lemma, $\overline{U}$ contains the aforementioned $z'$ and $\left(0, 0, \frac{b}{1-c}\right)$.
- Additionally, $\overline{U}$ either contains $(0, 0, e_i)$ for each $i$ or contains $(0, 0, e_1)$, $(0, 0, e_i)$ for some $i$ and $(0, 0, e_1 + e_i)$ for the remaining $i$.
- We apply an automorphism so that $\overline{U}$ contain $(0, 0, e_i)$ for each $i$. This automorphism sends $(0, 0, \frac{b}{1-c})$ to $(0, 0, b')$ such that $b'_i < 0$ and $\{1 - c, b'_1, \ldots, b'_n\}$ $\mathbb{Z}$ linearly independent.
- Then the results for $\mathbb{R}^n$ give us the theorem for $H_{mn}$. 