Hilbert Space Extensions of Anosov Diffeomorphisms

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Abstract
We consider extensions of Anosov diffeomorphisms on infranilmanifolds with Hilbert space fibers and disprove a conjecture regarding conditions equivalent to transitivity of such extensions.

1 Introduction
Many compact hyperbolic systems are transitive, as are nearby systems. A natural class of noncompact examples is that of skew products of Anosov diffeomorphisms on an infranilmanifold $T : X \to X$ over a Hilbert space $H$. Equipped with a Hölder function $f : X \to H$, define

$$
\hat{T}_f : X \times H \to X \times H
$$

$$
\hat{T}_f : (x, t) \mapsto (Tx, t + f(x))
$$

We define $\hat{T}_f$ to be stably transitive if, for nearby $f'$ in the $C^0$ topology, $\hat{T}_{f'}$ is transitive. We associate to each periodic orbit $T^kx = x$ a weight $f_k(x) := \sum_{i=0}^{k-1} f(T^ix)$, and denote $\mathcal{L}_f = \{ f_k(x) : T^kx = x \}$.

Lastly, we define the following useful hypothesis.

Definition A set $S$ in $H$ is said to satisfy the General Inseperability Hypothesis if $S$ is not contained on one size of some hyperplane.

With this, we may state the following theorem of Nitica and Pollicot on transitivity for $H = \mathbb{R}^n$ [1].

Theorem 1.1 Take $X$ an infranilmanifold and $T : X \to X$ an Anosov diffeomorphism. Then the following are equivalent for a Hölder function $f : X \to \mathbb{R}^n$:

1. $\mathcal{L}_f$ satisfies the General Inseperability Hypothesis;
2. $\hat{T}_f$ is transitive;
3. $\hat{T}_f$ is stably transitive;
4. There exist both positive and negative unbounded orbits in any direction (i.e., for any $v \in \mathbb{R}^n$ and $K > 0$, there exist $x, y \in X$ such that $T^kx = x, T^ky = y$ and $\langle f_k(x), v \rangle > K, \langle f_l(y), v \rangle < -K$).
The following theorem is the main theorem of this paper.

**Theorem 1.2** For a general Hilbert space $\mathcal{H}$, no two of the following three conditions are equivalent.

1. $\mathcal{L}_f$ satisfies the General Inseperability Hypothesis;
2. $\hat{T}_f$ is transitive;
3. $\hat{T}_f$ is stably transitive;

The inequivalence of (2) and (3) is proven in [2]; in particular, as stable transitivity clearly implies transitivity, $(2) \not\Rightarrow (3)$. As we also clearly have $(2) \Rightarrow (1)$, we cannot have (1) and (3) equivalent.

2 Constructing the Counterexample

**Proposition 2.1** The inseperability of $\mathcal{L}_f$ does not imply the transitivity of $\hat{T}_f$ for a general $T$, $f$, and $\mathcal{H}$.

**Proof** Take $X = T^2$, $\mathcal{H} = L^2_{[0,1]}$, $T = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$, and define $f$ as follows:

First, define $g(x) = \begin{cases} 4x, & \text{if } 0 \leq x < \frac{1}{4} \\ 2 - 4x, & \text{if } \frac{1}{4} \leq x < \frac{3}{4} \\ 4x - 4, & \text{if } \frac{3}{4} \leq x \leq 1. \end{cases}$

Then, define $f: T^2 \to L^2_{[0,1]}$ by

$$f: (x, y) \mapsto \text{sgn}(g(x))\chi_{[0,\lfloor g(x) \rfloor]} + \text{sgn}(g(y))\chi_{[0,\lfloor g(y) \rfloor]}.$$

It is easy to see that $f$ is Hölder with exponent $\frac{1}{2}$. In addition, $\text{Im}(f) \subseteq \{ v \in L^2_{[0,1]} \mid v \text{ is integer valued a.e.} \}$, a closed additive subgroup of $L^2_{[0,1]}$; as such, no point can have a dense orbit, and $\hat{T}_f$ fails to be transitive. We also have that $f(-x, -y) = -f(x, y)$ and $T(-x, -y) = -T(x, y)$, so that $\mathcal{L}_f = -\mathcal{L}_f$, so that we only need to prove that $\mathcal{L}_f$ is not contained in some hyperplane.

In a technical lemma contained in Section 3, we will construct a set $E$ dense in $[0, 1]$ such that $\{ \chi_{[0, e]} \mid e \in E \} \subseteq \text{Sp}(\mathcal{L}_f)$. The span of the left set is clearly dense in the set of simple functions, which is then dense in $L^2_{[0,1]}$; so upon taking the span and closure of both sides, we get $L^2_{[0,1]} \subseteq \text{Sp}(\mathcal{L}_f)$, and thus $L^2_{[0,1]} = \text{Sp}(\mathcal{L}_f)$. As a result, for any $\phi \in (L^2_{[0,1]})^*$ with $\phi(w) = 0$ for all $w \in \mathcal{L}_f$, we necessarily have $\phi = 0$. Thus $\mathcal{L}_f$ is contained in no hyperplane. 

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3 Fibonacci-Like Sequences Modulo $2^n$

Lemma 3.1 Let $n \geq 7$ be odd, and let $0 \leq k \leq 2^{n-2} - 1$. Then, there exists $0 \leq l \leq 2^{n-1} - 1$ such that $T^{3 \cdot 2^{n-2}} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and

$$3 \cdot 2^{n-2} \cdot \sum_{i=0}^{3 \cdot 2^{n-2} - 1} f \left( T^i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right) = 2^{3 \cdot 2^{n-1}} \text{sgn}(g(y_0)) \chi_{[0,|g(y_0)|]},$$

where $x_0 = \frac{2l+1}{2^n}$ and $y_0 = \frac{4k+2}{2^n}$.

Before beginning the proof of Lemma 3.1, we introduce a definition and make a remark about the form of $T$.

Definition Let $m \geq 2$ and let $s_0, s_1, \ldots$ be a sequence of integers modulo $m$. We say that $s_0, s_1, \ldots$ is a Fibonacci-like sequence modulo $m$ if

$$s_{i+2} \equiv s_{i+1} + s_i \pmod{m} \quad \forall i \geq 0.$$

A Fibonacci-like sequence is a sequence of integers satisfying the above recurrence relation with equality replacing congruence modulo $m$.

It is clear that any Fibonacci-like sequence is uniquely determined by any two adjacent terms. We denote by $F_0, F_1, \ldots$, the standard Fibonacci-like sequence beginning with the terms $F_0 = 0, F_1 = 1$.

We call $U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ the Fibonacci matrix, observing that if $s_0, s_1, \ldots$ is a Fibonacci-like sequence, then

$$U \begin{pmatrix} s_{i+1} \\ s_i \end{pmatrix} = \begin{pmatrix} s_{i+2} \\ s_{i+1} \end{pmatrix}.$$

By induction, it is easily seen that $\forall j \geq 1$,

$$U^j \begin{pmatrix} s_{i+1} \\ s_i \end{pmatrix} = \begin{pmatrix} s_{i+j+1} \\ s_{i+j} \end{pmatrix}.$$

These results also hold for Fibonacci-like sequences modulo $m$, if we replace equality by elementwise congruence modulo $m$. In particular, let $s_0, s_1, \ldots$ be a Fibonacci-like sequence modulo $2^n$ for some $n \geq 1$. Then, since $T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = U^2$, we have that

$$T \begin{pmatrix} s_{i+1} \\ s_i \end{pmatrix} \equiv \begin{pmatrix} s_{i+3} \\ s_{i+2} \end{pmatrix} \pmod{2^n},$$

and hence that

$$T \begin{pmatrix} \frac{2s_{i+1}}{2^n} \\ \frac{s_i}{2^n} \end{pmatrix} = \frac{1}{2^n} T \begin{pmatrix} s_{i+1} \\ s_i \end{pmatrix} \equiv \frac{1}{2^n} \begin{pmatrix} s_{i+3} \\ s_{i+2} \end{pmatrix} \pmod{1}.$$
Thus, Lemma 3.1 is actually a statement about the Fibonacci-like sequence modulo $2^n$ that contains the consecutive terms $4k+2, 2l+1$. The proof of Lemma 3.1 will proceed by a series of sublemmas that give properties of Fibonacci-like sequences modulo $2^n$. The techniques used here are in the spirit of those seen in [3].

**Sublemma 3.2** Let $n \geq 1$. Then, $U^{3 \cdot 2^{n-1}} \equiv I \pmod{2^n}$. If $n \geq 3$, then $U^{3 \cdot 2^{n-2}} \equiv (2^{n-1} + 1) I \pmod{2^n}$.

**Proof** We proceed by induction on $n$. Observe first that $\forall j \geq 1$,

$$U^j \begin{pmatrix} 1 \\ 0 \end{pmatrix} = U^j \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} F_{j+1} \\ F_j \end{pmatrix},$$

and

$$U^j \begin{pmatrix} 0 \\ 1 \end{pmatrix} = U^{j-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} F_j \\ F_{j-1} \end{pmatrix}.$$

This gives that

$$U^j = \begin{pmatrix} F_{j+1} & F_j \\ F_j & F_{j-1} \end{pmatrix}.$$ 

In particular,

$$U^3 = \begin{pmatrix} F_4 & F_3 \\ F_3 & F_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \equiv I \pmod{2}.$$

Now assume that $U^{3 \cdot 2^{n-1}} \equiv I \pmod{2^n}$ for some $n \geq 1$. It follows that $U^{3 \cdot 2^{n-1}} = I + 2^n C$ for some matrix $C$ with integer entries, and hence that

$$U^{3 \cdot 2^{n}} = (I + 2^n C)^2 = I + 2^{n+1} C + 2^{2n} C^2 \equiv I \pmod{2^{n+1}}.$$

Similarly, we see that

$$U^6 = \begin{pmatrix} F_7 & F_6 \\ F_6 & F_5 \end{pmatrix} = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix} \equiv 5I \pmod{8},$$

and if we assume that $U^{3 \cdot 2^{n-2}} \equiv (2^{n-1} + 1) I \pmod{2^n}$ for some $n \geq 3$, we have that $U^{3 \cdot 2^{n-2}} = (2^{n-1} + 1) I + 2^n C$ for some matrix $C$ with integer entries. This gives

$$U^{3 \cdot 2^{n-1}} = \left( (2^{n-1} + 1) I + 2^n C \right)^2 = (2^{2n-2} + 2^n + 1) I + (2^{n-1} + 1) 2^{n-1} C + 2^{2n} C^2 \equiv (2^n + 1) I \pmod{2^{n+1}},$$

using that $n \geq 3$ implies that $2^{n+1}$ divides $2^{2n-2}$. }

**Corollary 3.3** Let $n \geq 1$, and let $s_0, s_1, \ldots$ be a Fibonacci-like sequence modulo $2^n$ with at least one odd term. Then, $s_0, s_1, \ldots$ is periodic with minimal period $\delta(n) = 3 \cdot 2^{n-1}$. 

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Proof First note that if any two adjacent terms of a Fibonacci-like sequence are even, then the entire sequence is even. Then, by verifying that all possible pairs of integers modulo 2 involving an odd term appear as adjacent terms in the Fibonacci-like sequence, 0, 1, 1, 0, 1, 1, . . . , we see that the only Fibonacci-like sequences modulo 2 containing an odd term are shifts of the above sequence. Thus, all such sequences have minimal period 3. Similarly, any Fibonacci-like sequence modulo 4 containing an odd term is either a shift of the sequence 0, 1, 1, 2, 3, 1, 0, 1, . . . or a shift of the sequence 0, 3, 3, 2, 1, 3, 0, 3, . . . and hence has minimal period 6.

Now let \( n \geq 3 \) and assume that all Fibonacci-like sequences modulo \( 2^{n-1} \) that contain an odd term are periodic with minimal period \( 3 \cdot 2^{n-2} \). Let \( s_0, s_1, \ldots \) be a Fibonacci-like sequence modulo \( 2^n \) with \( s_j \) odd for some \( j \geq 0 \). By Sublemma 3.2, we have \( \forall i \geq 0 \)

\[
\begin{align*}
(s_{i+3} \cdot 2^{n-1} + 1) & \equiv U^3 2^n - (s_{i+1}) \equiv I (s_{i+1}) \pmod{2^n}.
\end{align*}
\]

Thus, \( s_0, s_1, \ldots \) is periodic and its minimal period \( \delta(n) \) must divide \( 3 \cdot 2^{n-1} \). Additionally, \( s_0, s_1, \ldots \) can also be viewed as Fibonacci-like sequence modulo \( 2^{n-1} \) that contains an odd term and hence has minimal period \( 3 \cdot 2^{n-2} \), when taken modulo \( 2^{n-1} \). This implies that \( 3 \cdot 2^{n-2} \) divides \( \delta(n) \). Therefore, \( \delta(n) = 3 \cdot 2^{n-1} \) or \( 3 \cdot 2^{n-2} \). However,

\[
\begin{align*}
(s_{j+3} \cdot 2^{n-2} + 1) & \equiv U^3 2^{n-2} - (s_{j+1}) \equiv (2^{n-1} + 1) I (s_{j+1}) \pmod{2^n}.
\end{align*}
\]

Since \( s_j \) is odd, we have that \( s_j = 2k + 1 \) for some \( k \in \mathbb{Z} \) and hence

\[
(2^{n-1}) s_j = 2^n k + 2^{n-1} \equiv 2^{n-1} \pmod{2^n}.
\]

Then,

\[
s_{j+3} \cdot 2^{n-2} \equiv (2^{n-1} + 1) s_j \equiv s_j + 2^{n-1} \equiv s_j \pmod{2^n},
\]

so that \( s_0, s_1, \ldots \) does not have period \( 3 \cdot 2^{n-2} \). This gives that \( \delta(n) = 3 \cdot 2^{n-1} \).

Corollary 3.4 Let \( n \geq 3 \), and let \( s_0, s_1, \ldots \) be a Fibonacci-like sequence modulo \( 2^n \). Then, \( \forall i \geq 0 \),

\[
s_{i+\delta(n-1)} \equiv \begin{cases} s_i, & \text{if } s_i \text{ is even} \\ s_i + 2^{n-1}, & \text{if } s_i \text{ is odd} \end{cases} \pmod{2^n}.
\]

Proof Note that \( \delta(n-1) = 3 \cdot 2^{n-2} \). The computation for \( s_i \) odd is seen in the previous proof. The computation for \( s_i \) even follows similarly.

Sublemma 3.5 Let \( n, j \in \mathbb{Z}^+ \) with \( n - j \geq 3 \). Then,

\[
U^{\delta(n-j-1)} \equiv \begin{pmatrix} (a + 2b) 2^n - 1 + 1 \\ b 2^n - 1 + 1 \\ a 2^n - 1 + 1 \end{pmatrix} \pmod{2^n}
\]

for some \( a, b \in \mathbb{Z} \) with \( a \) odd. Furthermore, for fixed \( j \), the same choice of \( a, b \) will satisfy the above congruence \( \forall n \geq 2j + 3 \).

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Proof Applying Sublemma 3.2, we see that
\[
U^{\delta(n-j-1)} \equiv \begin{pmatrix} 2^{n-j-1} + 1 & 0 \\ 0 & 2^{n-j-1} + 1 \end{pmatrix} \pmod{2^n}
\]
so that
\[
U^{\delta(n-j-1)} \equiv \begin{pmatrix} a_1 2^{n-j} + 2^{n-j-1} + 1 & a_2 2^{n-j} \\ a_3 2^{n-j} & a_4 2^{n-j} + 2^{n-j-1} + 1 \end{pmatrix} \pmod{2^n}.
\]
And since
\[
U^m = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} = \begin{pmatrix} F_m + F_{m-1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \forall m \geq 1,
\]
we see that the desired congruence holds for \(a = 2a_4 + 1\) (an odd number) and \(b = a_2\). We will see later that \(b\) is also odd.

To prove the second statement, let \(n \geq 2j + 3\) and \(a, b \in \mathbb{Z}\) with \(a\) odd and assume that
\[
U^{\delta(n-j-1)} \equiv \begin{pmatrix} (a + 2b) 2^{n-j-1} + 1 & b 2^{n-j} \\ b 2^{n-j} & a 2^{n-j-1} + 1 \end{pmatrix} \pmod{2^n}.
\]
Then, \(U^{\delta(n-j-1)} = B + 2^n C\), where \(B\) is the matrix on the right side of the above congruence and \(C\) is some matrix. Squaring both sides yields
\[
U^{\delta(n-j)} = B^2 + 2^n (BC + CB) + 2^n C^2.
\]
Additionally, noting that \(B\) has even off-diagonal entries and odd diagonal entries, we may write \(B = 2D + I\) for some matrix \(D\). Thus,
\[
BC + CB = (2D + I) C + C (2D + I) = 2 (C + DC + CD),
\]
This gives
\[
U^{\delta(n-j-1)} = B^2 + 2^n (C + DC + CD) + 2^n C^2
\]
\[
\equiv B^2 \pmod{2^{n+1}}
\]
\[
\equiv \begin{pmatrix} (a + 2b) 2^{n-j-1} + 1 & b 2^{n-j} + 1 \\ b 2^{n-j} + 1 & a 2^{n-j-1} + 1 \end{pmatrix} \pmod{2^{n+1}},
\]
where above we have used that \(n \geq 2j + 3\) implies that \(2^{n+1}\) divides \(2^{2n-2j-2}\), \(2^{2n-2j-1}\), and \(2^{2n-2j}\).

Corollary 3.6
\[
U^{\delta(n-2)} \equiv \begin{pmatrix} 2^{n-2} + 1 & 2^{n-1} \\ 2^{n-1} & -2^{n-2} + 1 \end{pmatrix} \pmod{2^n} \ \forall n \geq 5.
\]
\[
U^{\delta(n-3)} \equiv \begin{pmatrix} 2^{n-3} + 1 & 2^{n-2} \\ 2^{n-2} & -2^{n-3} + 1 \end{pmatrix} \pmod{2^n} \ \forall n \geq 7.
\]
Proof A straightforward calculation reveals that

\[ U^{\delta(3)} = U^{12} \equiv \begin{pmatrix} 9 & 16 \\ 16 & 25 \end{pmatrix} \pmod{32} \equiv \begin{pmatrix} 8 + 1 & 16 \\ 16 & -8 + 1 \end{pmatrix} \pmod{32}, \]

and that

\[ U^{\delta(4)} = U^{24} \equiv \begin{pmatrix} 17 & 32 \\ 32 & 113 \end{pmatrix} \pmod{128} \equiv \begin{pmatrix} 16 + 1 & 32 \\ 32 & -16 + 1 \end{pmatrix} \pmod{128}. \]

Applying Sublemma 3.5 with \( j = 1 \) and \( j = 2 \) gives the desired results. 

Remark Note that for \( n = 4 \), \( U^{\delta(n-2)} = U^n \equiv \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix} \pmod{128}. \) Thus, by Corollary 3.6, \( \forall n \geq 4 \), the off-diagonal entries of \( U^{\delta(n-2)} \) are congruent to \( 2^{n-1} \) modulo \( 2^n \). In the context of Sublemma 3.5, if \( n,j \in \mathbb{Z}^+ \) with \( n-j \geq 3 \), then the off-diagonal entries of \( U^{\delta(n-j-1)} \) are congruent to \( 2^{n-j} \) modulo \( 2^{n-j+1} \). Therefore, the coefficient \( b \) in Sublemma 3.5 is odd.

Corollary 3.7 Let \( n,j \in \mathbb{Z}^+ \) with \( n-j \geq 3 \), let \( s_0, s_1, \ldots \) be a Fibonacci-like sequence modulo \( 2^n \), let \( i \geq 0 \), and let \( a, b \in \mathbb{Z} \) odd satisfy the congruence in Sublemma 3.5. If \( s_i \equiv 2u \pmod{2^{i+1}} \) and \( s_{i+1} \equiv v \pmod{2^i} \) for \( u, v \in \mathbb{Z} \), then

\[ s_{i+\delta(n-j-1)} \equiv s_i + (au + bv) 2^{n-j} \pmod{2^n}. \]

In particular, if \( t_0, t_1, \ldots \) is a Fibonacci-like sequence modulo \( 2^n \), \( m \geq 0 \), and either

\[ \begin{cases} t_m \equiv 2^j + 2u \pmod{2^{i+1}} \\ t_{m+1} \equiv v \pmod{2^i} \end{cases} \]

or

\[ \begin{cases} t_m \equiv 2u \pmod{2^{i+1}} \\ t_{m+1} \equiv 2^{j-1} + v \pmod{2^j} \end{cases}, \]

then

\[ t_{m+\delta(n-j-1)} \equiv t_m + (au + bv) 2^{n-j} + 2^{n-1} \pmod{2^n}. \]

Proof For the first statement, we have

\[ \begin{pmatrix} s_{i+\delta(n-j-1)+1} \\ s_{i+\delta(n-j-1)} \end{pmatrix} \equiv U^{\delta(n-j-1)} \begin{pmatrix} s_{i+1} \\ s_i \end{pmatrix} \equiv U^{\delta(n-j-1)} \begin{pmatrix} 2^j + v \\ 2^{j+1}k + 2u \end{pmatrix} \pmod{2^n} \]

for some \( k, l \in \mathbb{Z} \). Then, applying Sublemma 3.5

\[ s_{i+\delta(n-j-1)+1} \equiv b2^{n-j} (2^j + V) + a2^{n-j-1} (2^{j+1}k + 2u) + s_i \pmod{2^n}, \]

\[ \equiv s_i + (au + bv) 2^{n-j} \pmod{2^n}. \]

The second statement follows easily from the first and the fact that \( a, b \) are odd.

Corollary 3.8 Let \( n \geq 4 \), and let \( s_0, s_1, \ldots \) be a Fibonacci-like sequence modulo \( 2^n \) that contains an odd term. Then, \( \forall i \geq 0 \),

\[ s_{i+\delta(n-2)} \equiv \begin{cases} s_i, & \text{if } s_i \equiv 2 \pmod{4} \\ s_i + 2^{n-1}, & \text{if } s_i \equiv 0 \pmod{4} \end{cases} \pmod{2^n}. \]
Proof If \( n \geq 5 \), then, as seen in Corollary 3.6, we have the \( j = 1 \) Sublemma 3.5 coefficients \( a = -1 \) and \( b = 1 \). If \( n=4 \), then \( a = 1 \) and \( b = -1 \) as seen in the remark above. In both cases, \( a + b = 0 \). Let \( i \geq 0 \) and suppose that \( s_i \equiv 2 \) (mod 4). As seen in the proof of Corollary 3.3, \( s_0, s_1, \ldots \) will reduce to a shift of the sequence 0, 1, 1, 0, \ldots when viewed modulo 2. In particular, any even term of \( s_0, s_1, \ldots \) must be followed by an odd term so that \( s_{i+1} \equiv 1 \) (mod 2). Then, applying Corollary 3.7 with \( j = 1 \) yields

\[
 s_{i+j(n-2)} \equiv s_i + (a \cdot 1 + b \cdot 1) 2^{n-1} \equiv s_i \pmod{2^n}.
\]

The case \( s_i \equiv 0 \) (mod 4) follows from the second statement of Corollary 3.7, noting that \( s_{i+1} \) must still be odd. \( \blacksquare \)

**Corollary 3.9** Let \( n \geq 7 \), and let \( s_0, s_1, \ldots \) be a Fibonacci-like sequence modulo \( 2^n \). Suppose that \( s_0, s_1, \ldots \) reduces to a shift of either of the Fibonacci-like sequences 4, 3, 7, 2, 1, 3, 4, 7, 3, 2, 5, 7, \ldots or 4, 1, 5, 6, 3, 1, 4, 5, 1, 6, 7, 5, \ldots when viewed modulo 8. Then, \( \forall i \geq 0 \) such that \( s_i \equiv 2 \) (mod 4),

\[
 s_{i+j(n-3)} \equiv s_i \pmod{2^n}.
\]

Proof \( n \geq 7 \), so as seen in Corollary 3.6, we have the \( j = 2 \) Sublemma 3.5 coefficients \( a = -1 \) and \( b = 1 \). Let \( i \geq 0 \) and assume first that \( s_i \equiv 2 \) (mod 8). Checking the two Fibonacci-like sequences modulo 8 in the hypotheses, we see that \( s_{i+1} \equiv 1 \) (mod 4). Then, applying Corollary 3.7 with \( j = 2 \) yields

\[
 s_{i+j(n-3)} \equiv s_i + (a \cdot 1 + b \cdot 1) 2^{n-1} \equiv s_i \pmod{2^n}.
\]

Similarly, if \( s_i \equiv 6 \) (mod 8), we have that \( s_{i+1} \equiv 3 \) (mod 4), and hence \( s_{i+j(n-3)} \equiv s_i \) (mod \( 2^n \)). \( \blacksquare \)

**Corollary 3.10** Let \( n \geq 7 \) odd, let \( 3 \leq j \leq \frac{n-1}{2} \), and let \( s_0, s_1, \ldots \) be a Fibonacci-like sequence modulo \( 2^n \). Suppose that \( s_{i+j(n-j)} \equiv s_i \) (mod \( 2^n \)) and \( s_i \) is even for some \( i \geq 0 \). Then,

\[
 s_{i+j(n-j-1)} \equiv s_i \text{ or } s_i + 2^{n-1} \equiv s_i \pmod{2^n}.
\]

Proof Observe first that \( j \leq \frac{n-1}{2} \) implies that \( j-1 \leq n-j-2 \) and hence \( \delta(j) = 3 \cdot 2^{j-1} \) divides \( \delta(n-j-1) = 3 \cdot 2^{n-j-2} \). Define \( k \) such that \( \delta(n-j-1) = k \delta(j) \). Clearly,

\[
 s_{i+j(n-j-1)+1} = s_{i+k \delta(j)+1} \equiv s_{i+1} \pmod{2^j},
\]

since \( \delta(j) \) is the period of the sequence modulo \( 2^j \). Furthermore, since \( s_i \) is even, by \( k \) applications of Corollary 3.4 with \( j+1 \) taking the place of \( n \), we have that \( s_{i+j(n-j-1)} \equiv s_i \) (mod \( 2^{j+1} \)). Noting that \( n-j \geq \frac{n+1}{2} \geq 4 \), we may apply the first statement of Corollary 3.7 to obtain \( C \in \mathbb{Z} \) such that

\[
 s_{i+j(n-j-1)} \equiv s_i + C \pmod{2^n}
\]

and

\[
 s_{i+2 \delta(n-j-1)} \equiv s_{i+j(n-j-1)} + C \pmod{2^n}.
\]

Then,

\[
 s_{i+j(n-j)} = s_{i+2 \delta(n-j-1)} \equiv s_i + 2C \pmod{2^n}.
\]

Combining this result with the assumption that \( s_{i+j(n-j)} \equiv s_i \) (mod \( 2^n \)) gives that \( C \equiv 0 \) or \( 2^{n-1} \) (mod \( 2^n \)). \( \blacksquare \)
Sublemma 3.11 Let \( n \geq 7 \) be odd, and let \( 0 \leq k \leq 2^{n-2} - 1 \). Then, there exists \( l \in \mathbb{Z} \) such that if \( s_0, s_1, \ldots \) is the Fibonacci-like sequence modulo \( 2^n \) generated by \( s_0 = 4k + 2 \) and \( s_1 = 2l + 1 \), then for all \( 2 \leq j \leq \frac{n-1}{2} \),
\[
s_{\delta(n-j-1)} \equiv s_0 \pmod{2^n}.
\]

Proof We proceed by induction on the upper bound for \( j \). The case \( j = 2 \) follows from Corollary 3.9: if \( s_0 \equiv 2 \pmod{8} \), take \( l = 0 \); if \( s_0 \equiv 6 \pmod{8} \), take \( l = 1 \). In either case, \( s_0, s_1, \ldots \) will reduce to a shift of one the sequences in the hypotheses of Corollary 3.9 when viewed modulo 8.

Now, let \( 3 \leq N \leq \frac{n-1}{2} \), and assume that \( \exists l \in \mathbb{Z} \) such that
\[
s_{\delta(n-j-1)} \equiv s_0 \pmod{2^n} \quad \text{for } 2 \leq j \leq N - 1.
\]
In particular, \( s_{\delta(n-N)} \equiv s_0 \pmod{2^n} \) and \( s_0 \) is even. Thus, we may apply Corollary 3.10 with \( N \) taking the place of \( j \) and \( i = 0 \) to obtain
\[
s_{\delta(n-N-1)} \equiv s_0 + C \pmod{2^n}, \quad \text{where } C = 0 \text{ or } 2^{n-1}.
\]
If \( C = 0 \), we are finished. If \( C = 2^{n-1} \), by the second statement of Corollary 3.7, we have
\[
t_{\delta(n-N-1)} \equiv t_0 \pmod{2^n},
\]
where \( t_0, t_1, \ldots \) is the Fibonacci-like sequence sequence modulo \( 2^n \) generated by \( t_0 = s_0 = 4k + 2 \) and \( t_1 = s_1 + 2^{N-1} = 2 \left( l + 2^{N-2} \right) + 1 \).

Then, \( t_1 \equiv s_1 \pmod{2^{N-1}} \), and moreover \( t_l \equiv s_l \pmod{2^i} \) for all \( 2 \leq j \leq N - 1 \), since \( 2^j \) divides \( 2^{N-1} \) for all such \( j \). Applying the first statement of Corollary 3.7 yields
\[
t_{\delta(n-j-1)} \equiv t_0 \pmod{2^n} \quad \text{for } 2 \leq j \leq N - 1.
\]

Sublemma 3.12 Let \( n \geq 7 \) be odd, and let \( 0 \leq k \leq 2^{n-2} - 1 \). Then, there exists \( l \in \mathbb{Z} \) such that if \( s_0, s_1, \ldots \) is the Fibonacci-like sequence modulo \( 2^n \) generated by \( s_0 = 4k + 2 \) and \( s_1 = 2l + 1 \), then for all \( 3 \leq j \leq \frac{n-1}{2} \) and \( i \geq 0 \),
\[
s_{i+\delta(n-j-1)} \equiv \begin{cases} s_i, & \text{if } i \equiv 0 \pmod{\delta(j)} \\ s_i + 2^{n-1}, & \text{if } i \equiv \delta(j-1) \pmod{\delta(j)} \end{cases} \pmod{2^n}.
\]

Proof Take \( l \in \mathbb{Z} \) satisfying Sublemma 3.11, and let \( 3 \leq j \leq \frac{n-1}{2} \). We will prove the following:

Claim Let \( i \equiv 0 \pmod{\delta(j-1)} \) and suppose that \( s_{i+\delta(n-j-1)} \equiv s_i + C \) where \( C = 0 \) or \( 2^{n-1} \). Then,
\[
s_{i+\delta(j-1)+\delta(n-j-1)} \equiv s_{i+\delta(j-1)} + C + 2^{n-1}.
\]
It is easy to see that this claim and Sublemma 3.11 give the desired result.
Proof of Claim $j \geq 3$ implies that $\delta(2)$, the period of the sequence viewed modulo 4, divides $\delta(j-1)$, and hence that $s_i \equiv s_0 \equiv 2 \pmod{4}$. Since $j+1 \geq 4$, we may apply Corollary 3.8 with $j+1$ taking the place of $n$ to obtain that $s_i + \delta(j-2) \equiv s_i \pmod{2^{j+1}}$. Similarly, $s_{i+1} \equiv s_1 \equiv 1 \pmod{2}$, and since $j \geq 3$, we may apply Corollary 3.4 with $j$ taking the place of $n$ to obtain that $s_{i+1} + \delta(n-j+1) \equiv s_{i+1} + 2^{j-1} \pmod{2}$. Then, the second statement of Corollary 3.7 yields

$$s_{i+\delta(j-1)+\delta(n-j)} \equiv s_i + \delta(j-1) + C + 2^{n-1} \pmod{2^n}.$$ 

Sublemma 3.13 Let $n, j \in \mathbb{Z}^+$ with $n-j \geq 3$, let $s_0, s_1, \ldots$ be a Fibonacci-like sequence modulo $2^n$, and let $T = U^2$.

We are now prepared to prove Lemma 3.1.

Proof of Lemma 3.1 Take $l \in \mathbb{Z}$ satisfying Sublemma 3.12, and denote by $s_0, s_1, \ldots$ the Fibonacci-like sequence modulo $2^n$ generated by $s_0 = 4k + 2$ and $s_1 = 2l + 1$. It is clear that if $0 \leq s_i < 2^n$, then $0 \leq l \leq 2^{n-1} - 1$. Then, $T^a - T^{2n-a} = \left(\frac{x_0}{y_0}\right)$ follows from $T = U^2$ and Corollary 3.3. Additionally, from the form of $f$ and the action of $T = U^2$ on Fibonacci-like sequences, we have

$$\sum_{i=0}^{2^{n-2} - 1} f \left( T^i \left( \frac{x_0}{y_0} \right) \right) = \sum_{i=0}^{\delta(n)-2} f \left( T^i \left( \frac{n}{2^n} \right) \right) = \sum_{i=0}^{\delta(n)-1} \text{sgn} \left( g \left( \frac{n}{2^n} \right) \right) \chi[0, |g(s_i/2^n)|].$$

First, note that if $j = \frac{n-1}{2}$, then $j = n - j - 1$. Thus, the $j = \frac{n-1}{2}$ case of Sublemma 3.12, in fact gives that $s_i \equiv s_0 \pmod{2^n} \forall i \equiv 0 \pmod{\delta \left( \frac{n-1}{2} \right)}$, and hence that $g \left( \frac{n}{2^n} \right) = g \left( \frac{n}{2^n} \right)$ for all such $i$. There are $\delta(n)/\delta \left( \frac{n-1}{2} \right) = 2^{n-2}$ terms involving these values of $g$ in the above sum. It remains to check that all the other terms sum to zero. We will in fact show that all other terms cancel in pairs.

A simple computation reveals that for any $x \in \mathbb{R}/\mathbb{Z}$, $g(x + 1/2) + g(x) = 0$. Thus, if $s_i \equiv s_m + 2^{n-1} \pmod{2^n}$, then

$$\text{sgn} \left( g \left( \frac{s_i}{2^n} \right) \right) \chi[0, |g(s_i/2^n)|] + \text{sgn} \left( g \left( \frac{s_m}{2^n} \right) \right) \chi[0, |g(s_m/2^n)|] = 0.$$ 

For convenience, we say that $s_i$ cancels $s_m$ in this case. By Corollary 3.4, any odd $s_i$ will cancel $s_i + \delta(n-1)$, its counterpart a half-period away. Similarly, by Corollary 3.8, any $s_i \equiv 2 \pmod{4}$ in the first quarter-period will cancel $s_i + \delta(n-2)$, its counterpart in the second quarter-period. And since $n \geq 4$, $\delta(2)$ divides $\delta(n-2)$, the first and second quarter-periods are identical when viewed modulo 4 so that this cancels all $s_m \equiv 2 \pmod{4}$ in the second quarter-period. Analogous cancellation occurs between the third and fourth quarter-periods. The remaining terms $s_i$ will satisfy $s_i \equiv 2 \pmod{4}$ with $i \neq 0 \pmod{\delta \left( \frac{n-1}{2} \right)}$. 

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Let such an $s_i$ be given. Since all Fibonacci-like sequences modulo 4 with an odd term contain exactly one term congruent to 2 (mod 4) per period $\delta(2)$ and $s_0 \equiv 0 \pmod{4}$, we have that $i \equiv 0 \pmod{\delta(2)}$. Let $j$ be minimal such that $i \not\equiv 0 \pmod{\delta(j)}$. Clearly, $3 \leq j \leq \frac{n-1}{2}$ and by the minimality of $j$, $i \equiv \delta(j-1) \pmod{\delta(n-j)}$. Then, using a similar argument as above and Sublemma 3.12, we can obtain cancellation of all such $s_i$. □

The dense set $E \subseteq [0,1]$ required for Section 2 is given by

$$E = \left\{ \left\lfloor g\left(\frac{4k+2}{2^n}\right) \right\rfloor \mid n \geq 7 \text{ odd}, 0 \leq k \leq 2^{n-2} - 1 \right\}.$$

References

