The Frog Problem

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Preliminaries

- Cycle graph, such that each frog has two neighbors
- Identical probability $p$ for each frog at each iteration
- Restriction: A frog returns if it is more than one level above its neighbors
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Represent Motzkin Path as a vector

We call this a state

The number of states are enumerated by the central trinomial coefficients

Our goal is to get a bound for the asymptotic speed $S(K)$
Markov Chains

Definition

A **Markov Chain** is collection of random variables $X_t$, $t = 0, 1, ...$, having the property that, given the present, the future is conditionally independent of the past.
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- For any finite $K$ we have a Markov chain where the random variable represents transition probabilities between states.
Example, $K = 2$

\[
M = \begin{pmatrix}
1 & 2 & 3 \\
1 & p_{11} & p_{12} & p_{13} \\
2 & p_{21} & p_{22} & p_{23} \\
3 & p_{31} & p_{32} & p_{33}
\end{pmatrix}
\]
Example, $K = 2$

For $K = 2$ we have

$$Q = \begin{pmatrix} q^2 & pq & pq \\ 0 & pq + q^2 & 0 \\ 0 & 0 & pq + q^2 \end{pmatrix}$$

and

$$R = \begin{pmatrix} p^2 & 0 & 0 \\ pq & p^2 & 0 \\ pq & 0 & p^2 \end{pmatrix}.$$
Example, $K = 2$

It can be easily checked that

$$(I - Q)^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2p - 3}{p(p - 2)} \\ \frac{1}{2p(1 - p)} \\ \frac{1}{2p(1 - p)} \end{pmatrix}$$
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- We can do this for any fixed $K \in \mathbb{N}$, i.e. $S(K)$ exists
- In this example we see that the expected time spent in state 2 and state 3 is the same
- e.g. When $p = \frac{1}{2}$ we have $\begin{pmatrix} 8/3 \\ 2 \\ 2 \end{pmatrix}$
We need a new approach for when $K \to \infty$. Define

- $X_{n,k}$ are our random variables indexed by level and frog
- Indicator function $L_n(i,j)$
- $T(N,K) = \max_{k_1,\ldots,k_N} \{(\prod_{n=2}^{N} L_n(k_{n-1}, k_n)) \sum_{n=1}^{N} X_{n,k_n}\}$
- $r = E[\sum_{j=1}^{K} L_n(i,j)]$ for all $i$ and $n$
Theorem (Chang and Nelson)

If the moment generating function of the time it takes for a frog to jump, $X_{n,k}$, exists for some finite $0 < \theta_0$

$$\phi(\theta) \overset{\text{def}}{=} E[e^{\theta X_{n,k}}] < \infty \quad \text{for } \theta \leq \theta_0$$

then the asymptotic speed $S(K)$ for all of the frogs to jump higher than some fixed level is bounded below $\frac{1}{t^*}$, where

$$t^* = \inf\{t \geq 1 | rm(t) < 1\}$$

and

$$m(t) = \inf_{0<\theta<\theta_0} \{e^{-\theta t} \phi(\theta)\}.$$
Sketch of Proof

Definition

A Martingale is a sequence of random variables $X_1, X_2, \ldots$ where the following is true

$$E[|X_n|] < \infty \quad \text{and} \quad E[X_{n+1}|X_1, \ldots X_n] = X_n$$

$$M_n(\theta) = \frac{1}{(r\phi(\theta))^n} \sum_{k_1=1}^{K} \ldots \sum_{k_N=1}^{K} \left( \prod_{m=2}^{n} L_{m-1}(k_{m-1}, k_m) \right) e^{\theta \sum_{m=1}^{n} X_{m,k_m}}$$
Sketch of Proof

Lemma

For the system as previously defined,

\[ \frac{1}{S(K)} = \limsup_{N \to \infty} \frac{T(N, K)}{N} \leq t^*, \quad \text{a.s.} \]

\[ e^{\theta T(N, K)} = \max_{k_1, \ldots, k_N} \left\{ \prod_{n=2}^{N} L_n(k_{n-1}, k_n) e^{\theta \sum_{n=1}^{N} X_{n, k_n}} \right\} \leq (r\phi(\theta))^N M_N(\theta). \]
Sketch of Proof

Markov’s Inequality

\[ P(X > t) \leq \frac{E[X]}{t} \]

\[
P \left( \frac{T(N, K)}{N} > t \right) = P \left( e^{\theta T(N, K)} > e^{\theta N t} \right)
\]

\[
\leq e^{-\theta N t} E \left[ e^{\theta T(N, K)} \right]
\]

\[
\leq \frac{K}{r} \left( re^{-\theta t} \phi(\theta) \right)^N
\]
Sketch of Proof

Thus, since $\sum_N P \left( \frac{T(N,K)}{N} > t \right) < \infty$ if $rm(t) < 1$. Hence we have

$$P \left( \limsup_{N \to \infty} \left( \frac{T(N,K)}{N} > t \right) \right) = 0$$

which implies

$$\limsup_{n \to \infty} \frac{T(N,K)}{N} \leq t^*, \text{ a.s}$$
Main Theorem

Theorem

Given $K^2$ iterations, the probability that all frogs have jumped above level $N$, where $N = aK^2$ and $a < \frac{-\ln q}{\ln r}$, is $1 - \alpha \beta^{K^2}$, where $\beta < 1$. That is,

$$P(T(N, K) \leq K^2) \xrightarrow{K \to \infty} 1$$
Theorem

Given $K^2$ iterations, the probability that all frogs have jumped above level $N$, where $N = aK^2$ and $a < \frac{-\ln q}{\ln r}$, is $1 - \alpha \beta^{K^2}$, where $\beta < 1$. That is,

$$P(T(N, K) \leq K^2) \xrightarrow{K \to \infty} 1$$

$$P(T(N, K) > K^2) = P(e^{\theta T(N, K)} > e^{\theta K^2})$$

$$\phi(\theta) = \frac{pe^\theta}{1 - qe^\theta}$$
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