Symplectic Topology and Area-Preserving Maps of $S^2$

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Motivation

Question

Given a fixed area $A > 0$, is there a constant $\delta(A) > 0$, such that for any homeomorphism $f : S^2 \rightarrow S^2$ which preserves area and displaces a subset $E \subset S^2$ with area $A$, there is an $x \in S^2$ with $|f(x) - x| \geq \delta(A)$? What conditions are needed on $E$?
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Answer: Yes, but this is not true in the plane or on the torus.
Motivated by Hamilton’s equations in phase space coordinates \((p, q)\):

\[
\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}
\]

More generally,

**Definition**

A symplectic vector space is a real vector space with a bilinear form \(\omega\): \(V \times V \rightarrow \mathbb{R}\) that is skew-symmetric and nondegenerate.

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\omega(v, w) = -\omega(w, v), \quad \omega(v, w) = 0 \quad \forall w \iff v = 0
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The standard basis of $\mathbb{R}^{2n}$ given by $e_1, \ldots, e_n, f_1, \ldots, f_n$ where $f_i = e_{n+i}$ forms a \textit{symplectic basis}, that is,

$$\omega_o(e_i, e_j) = \omega_o(f_i, f_j) = 0, \quad \omega_o(e_i, f_j) = \delta_{ij}$$
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**Theorem**

For any symplectic vector space $(V, \omega)$ of dimension $2n$ there exists a symplectic basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ and a vector space isomorphism $\Psi : \mathbb{R}^{2n} \to V$ such that $\Psi^* \omega = \omega_o$. 

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- $M$ is even dimensional since each $T_xM$ is a symplectic vector space.

- $M$ is orientable. Why?

- If $M$ is closed, $H_2(M, \mathbb{R}) \neq 0$. Why? This means $S^n$ is not a symplectic manifold for $n \neq 2$. However, $CP^n$ all have symplectic structure.

The classic example is the cotangent bundle $T^*N$ of an $n$-dimensional smooth manifold $N$. We define coordinates in $T^*N$ by letting $q$ be choice of $n$ local coordinates in $N$ and then a 1-form on the tangent space has $n$ components $p$. In these local coordinates, the natural 2-form is $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$. 

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In fact, every symplectic manifold has this local structure:

**Theorem (Darboux)**

Let $(M, \omega)$ be a symplectic manifold. For each $p \in M$ there exists local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in an open neighborhood $U$ of $p$ such that $\omega|_U = \sum_{i=1}^n dx_i \wedge dy_i$. 
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There are no local invariants in symplectic geometry! This is much different than Riemannian geometry where the curvature is a local invariant.
Global Invariants

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**Theorem (Gromov Nonsqueezing)**

Let $Z^{2n}(R) = \{(x, y) \in \mathbb{R}^{2n} : x_1^2 + y_1^2 < R^2\}$ and $B^{2n}(r)$ be an open ball in $\mathbb{R}^{2n}$. If there is a symplectic embedding $f : B^{2n}(r) \hookrightarrow Z^{2n}(R)$ then $r \leq R$. 
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**Theorem (Gromov Nonsqueezing)**

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This theorem motivates a finer global invariant:

**Definition**

A symplectic capacity is a functor that assigns a number, \( c(M, \omega) \in [0, \infty] \), to a symplectic manifold \((M, \omega)\) such that

1. *(monotonicity)* If \( \text{dim}(M_1) = \text{dim}(M_2) \) and there is a symplectic embedding \( f : M_1 \hookrightarrow M_2 \) then \( c(M_1, \omega_1) \leq c(M_2, \omega_2) \)

2. *(conformality)* \( c(M, \lambda \omega) = |\lambda| c(M, \omega) \) for all \( \lambda \in \mathbb{R} \setminus \{0\} \)

3. *(strong nontriviality)* \( c(B^{2n}(1), \omega_o) = \pi = c(Z^{2n}(1), \omega_o) \)
The existence of a symplectic capacity is equivalent to the Nonsqueezing Theorem since it allows us to define the Gromov width:

$$w_G(M, \omega) = \sup_{r > 0} \{ \pi r^2 : B^{2n}(r) \text{ can be symplectically embedded in } M \}$$
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- H. Hofer and E. Zehnder introduced a special capacity for which existence can be proved without using Gromov Nonsqueezing, but we are not quite ready.
Hamiltonians

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**Definition**

A (time-dependent) Hamiltonian is a smooth function $H : [0, 1] \times M \to \mathbb{R}$. We denote $H_t = H(t, \cdot)$. 

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Conservation of Energy

Let $\eta \in X_t$ then $dH_t (\eta) = \omega (\eta, i^{-1} \omega (dH_t)) = \omega (\eta, \eta) = 0$. 

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A Hamiltonian diffeomorphism is the time-1 map of a Hamiltonian flow. Denote the group of compactly supported Hamiltonian diffeomorphisms as \( \text{Ham}_c(M, \omega) \). The Hofer norm on \( \text{Ham}_c(M, \omega) \) is defined as

\[ \| \psi \| = \inf_H \left\{ \int_0^1 \| H \| dt : \psi = \phi^1_H \right\} \]
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The Hofer-Zehnder capacity mentioned earlier is defined by

$$c_{HZ}(M, \omega) = \sup\{ \| H_t \| : H_t \text{ has no nonconstant periodic orbits with period } \leq 1 \}$$
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Hofer defined another capacity for open $U \subset \mathbb{R}^{2n}$ called the *displacement energy*

$$e(U) = \inf_{\psi \in Ham_c(\mathbb{R}^{2n}, \omega_0)} \{ \| \psi \| : \psi(U) \cap U = \emptyset \}$$
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and the $C^0$ continuity of the Hofer norm

$$\|\psi\| \leq CD\|\psi\|_{C^0} = CD \sup_x \|\psi(x) - x\|$$

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Hofer’s Geometry

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where \( D = \text{diameter supp}(\psi) \). And we finally have our first results:

**Theorem**

There exists \( \delta(A, D) > 0 \) such that for any \( \psi \in \text{Ham}_c(\mathbb{R}^{2n}, \omega_0) \) with \( D = \text{diameter supp}(\psi) \) that displaces an open set \( U \subset \mathbb{R}^{2n} \) with \( c_{HZ}(U) = A > 0 \), we have \( \|x - \psi(x)\| \geq \delta(A, D) \) for some \( x \in \mathbb{R}^{2n} \).

**Corollary**

There exists \( \delta(A, D) > 0 \) such that for any \( \psi \in \text{Symp}_c(\mathbb{R}^2, \omega_0) \) with \( D = \text{diameter supp}(\psi) \) that displaces an open path connected set \( U \subset \mathbb{R}^2 \) with \( \text{Area}(U) = A > 0 \), we have \( \|x - \psi(x)\| \geq \delta(A, D) \) for some \( x \in \mathbb{R}^2 \).
Spectral Norm

Unfortunately, the Hofer norm is not $C^0$-continuous on closed manifolds. Thus, to get our intended result we had to use the spectral norm $\gamma$ on $\text{Ham}(M,\omega)$ introduced by Y. G. Oh and introduced on spectral invariants of Floer Homology theory.
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From these we obtain

**Theorem**

Let $(M, \omega)$ be a closed 2-dimensional symplectic manifold. There exists $\delta(A) > 0$ such that for any $\phi \in Ham(M, \omega)$ that displaces an open set $U \subset M$ with $c_{HZ}(U) = A > 0$, we have $\|x - \phi(x)\| \geq \delta(A)$ for some $x \in M$. 
Using simply connectedness of $S^2$, connectedness of its symplectomorphism group, and a uniform approximation theorem of Y. G. Oh we obtain

**Theorem**

Let $(S^2, \omega_0)$ be the sphere with the canonical two-form. There exists $\delta(A) > 0$ such that for any area-preserving homeomorphism $f : S^2 \to S^2$ that displaces the closure of an open path connected set $U \subset S^2$ with $\text{Area}(U) = A > 0$, we have $\|x - f(x)\| \geq \delta(A)$ for some $x \in S^2$. 
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Then, by using the lifting theorem of continuous maps to the universal cover we have

**Corollary**

There exists $\delta(A) > 0$ such that for any area-preserving homeomorphism $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ that displaces the closure of an open path connected set $U \subset \mathbb{RP}^2$ with area $A$, we have $\|x - f(x)\| \geq \delta(A)$ for some $x \in \mathbb{RP}^2$. 
Questions???