BENDING FIELDS, RIGIDITY, AND THE BELLOWS CONJECTURE

TOÀN PHAN AND CASEY BUSH

Abstract. Aleksandrov and Pogorelov used bending fields (the velocity fields of continuous isometric deformations) to show infinitesimal rigidity of some classes of surfaces ([2] and [12]). Bending fields (and the rigidity matrix) were also successfully employed in the study of infinitesimal rigidity of polyhedra and tensegrity frameworks (see [7]). We will use these tools to re-derive a previously known result that almost all spherical polyhedra are rigid, and to finding the rate of change in volume of a flexible polyhedron during a flex. The latter leads us to two claims whose validity would imply the bellows conjecture. The first involves the bending vectors at the polyhedron's vertices, and is very elementary (see claim 4.3). The second involves a linear system whose matrix is the rigidity matrix (see claim 4.7). This seems to be a rather natural approach to proving the bellows conjecture that the volume of a polyhedron is constant under flexing.

Contents

Date: August 17, 2004.

We are indebted to Sergei Tabachnikov for his brilliant suggestions and his hospitality during my summer REU at the Pennsylvania State University. We would also like to thank Misha Guysinsky and Mohammad Ghomi for their support. Last but not least, we would like to thank our research fellows, in particular Mitka Vaintrob and Christopher Culter, for the helpful late night discussions, and for making us laugh. This research was carried out as part of the REU program at the Pennsylvania State University, summer 2004.

1. Prelude

1.1. History of the Rigidity Problem. For hundreds of years, mathematicians and engineers have been studying models of closed surfaces, and especially polyhedral models whose faces are made out of cardboard or metal plates and adjacent faces are hinged together. Most models of closed surfaces seemed to be rigid. If a model has an apparent flex, the flex had been traceable to distortion of the faces or the hinges. Euler conjectured in 1766 that all closed surfaces are rigid and in a letter to Lagrange in 1770 ([10]).

Cauchy was the first hero in this field, as he was in many others. He proved in 1813 that if two three-dimensional convex polyhedra have pairwise congruent faces and the faces are assembled in the same way, then the two polyhedra are congruent themselves. This immediately implies every convex polyhedron is rigid. We will briefly discuss Cauchy's (corrected) proof in section 3.1. Similar results with different approaches had been obtained for analytic surfaces by Liebmann in 1899 and Cohn-Vossen in 1936. Later Herglotz had a very succint proof of Cohn-Vossen's Theorem, and the surface needs to only have continuous second derivatives instead of analyticity. The reader can find a nice treatment of Cohn-Vossen's theorem in Hicks's book [11]. In 1975, Gluck showed that almost all closed simply connected closed surfaces are rigid (see [10]). His argument for the polyhedral case is very simple, and was re-derived many times (in fact we re-derived it while working with the rigidity matrix). Please refer to Connelly's extensive survey article on rigidity [7] for more information.

In spite of Cauchy's proof of the rigidity of convex polyhedra, no result was known for the non-convex case for a long time. The first embedded flexible non-convex polyhedron was discovered by Connelly in 1977 after years of searching. After Connelly, mathematicians have found some simpler flexible polyhedra. The simplest known flexible polyhedron was constructed by Klaus Steffen, and has only 9 vertices. Please see [8]. However, no example of a flexible closed surface has been found.

1.2. Volume of a Polyhedron under a Flex. It is an intriguing fact that the volume of a flexible polyhedron does not change when it is being flexed. The fact was nicknamed the *bellows conjecture* (because it says that there is no exact mathematical bellows). It was not until recently that the conjecture was proven and nicely presented in a joint paper between Sabitov, Connelly and Walz ([5]). Their idea was to show that the volume of a polyhedron is a root of a polynomial, which depends only on the edge lengths and the combinatorial type of the polyhedron. In this paper, we want to find a different method for proving the bellows conjecture by applying the notion of bending fields.

It is worthy to note that it has been proved the total mean curvature of a polyhedron is invariant under flexing [4]. Connelly also conjectured that a polyhedron will stay equidecomposable to itself under flexing. We will briefly discuss equidecomposability and the Dehn invariant in the last section.

1.3. Our Approach. There is a very useful tool for the rigidity problem of smooth surfaces: the so-called *bending fields* (studied in, for example, [2] and [12]). Our idea is to apply the theory of bending field to polyhedra. This approach not only helped us trace out the path people might have gone to define and prove the infinitesimal version of Cauchy's rigidity theorem (section 3.2), but also gave us a natural approach to the bellows conjecture.

A bending field of a surface is simply the velocity field at time $t = 0$ of a deformation of the surface. We will elaborate this notion in subsection 2.1. Aleksandrov found necessary conditions for a vector field on a certain surface to be a bending field. In our case of a polyhedron, the situation is quite simpler. The bending field associated with a flex must be linear on each face of the polyhedron. Therefore the bending field is spanned by the bending vectors at the vertices of the polyhedron. We give a simple necessary condition for a vector field on a polyhedron to be a flexing field in 2.2. This condition is given in terms of the field vectors at the vertices of the polyhedron, and is quite simple.

In section 4 we apply the notion of bending field to find the rate of change of the volume of a polyhedron under a flex. We give a formula in terms of the bending vectors at the vertices, the area of the faces and the unit outward normal vectors at the faces. These formula lead us to two conjectures whose validity implies the bellows conjecture. The first form involves the bending vectors at the vertices. The second only involves the vectors whose ends are at the vertices of the polyhedron. These forms are pretty simple. However, we have yet found a proof.

2. Bending Field

2.1. Bending Field of a Surface.

A vector field on a surface is called a *deforming field* if there exists a deformation f_t of the surface such that the field vector at any point q on the surface equals $d(f_t(q))/dt|_{t=0}$, the derivative at time $t = 0$ of the path $q(t) = f_t(q)$ that q travels during the deformation.

We are only concerned with *isometric deformations* or *bending* of a surface. A classic example is the isometric deformation from a helicoid to a catenoid. Please refer to [14] for a nice picture.

An *infinitesimal bending* of a surface in $\mathbb{R}^3$ is a deformation under which the length of any rectifiable curve on the surface is stationary. Precisely, if f_t is the deformation of a surface S ($t \in [0, 1)$ and $f_0 =$ Identity), we require that for any rectifiable curve γ on S, the rate of change of its length at time $t = 0$ be 0: $dl(f_t(\gamma))/dt|_{t=0} = 0$. Basically, this means f_t is isometric locally at $t = 0$.

Definition 2.1. A vector field on a surface S is called a *bending field* of S if it is the deforming field of an infinitesimal bending of S.

Example 2.2. In $\mathbb{R}^3$ with the usual Cartesian coordinate system, let us consider the plane $z = 0$. This plane has a bending field $B((x, y, 0)) = (0, 0, y)$. This bending field is associated with an infinitesimal bending $f_t((x, y, 0)) = (x, y \cos t, y \sin t)$ (rotation around the x-axis). See figure 2.1

Remark 2.3. An infinitesimal bending associated with a bending field does not need to be an isometry. It only needs to be isometric in a neighborhood of $t = 0$. For example, one can easily check $g_t((x, y, 0)) := (x, y, 0) + t(0, 0, y)$ is an infinitesimal bending. But it is not an isometric deformation.

However, in this paper, most bending fields are associated with some isometric deformations. We will furthermore mostly restrict our attention to isometric deformations of planes (faces of a polyhedra), and such an isometric deformation is

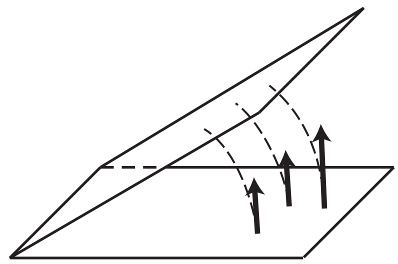

FIGURE 2.1. The bending field in example 2.2 is represented by the arrows.

simply a continuous family of isometric affine transformations. Then the bending field will always be a linear vector field as in example 2.2. The bending fields under these restrictions suffice our purpose of studying flexing of polyhedra. We will discuss this type of 'linear bending field' (or flexing field) in the next section.

2.2. **Flexing Field.**
A (continuous) deformation of a polyhedron/plane is called a *flex* if under that deformation every face of the polyhedron is included in a family of congruent faces. In other words, under a flex, every face undergoes a deformation consisting of a continuous family of isometric affine transformations in three-space. Informally, one can think of deforming a polyhedron whose faces are metal plates and the edges are hinges between the plates. A flex is automatically a bending (isometric deformation), and thus has an associated bending field, which we shall call a *flexing field*.

As we said above, under a flex, a single face is continuously deformed by a one-parameter family of isometric affine transformations (or an 'isometric affine homotopy'). Explicitly, if f_t, with $t \in [0,1)$, denotes the continuous deformation, then for some isometric affine transformation T_t^F on F, $f_t(p) = T_t^F(p)$ for all points p on a face F.

Since T_t^F is an isometric affine transformation, $T_t^F(p) = A_t^F p + b_t^F$ for some isometric linear transformation A_t^F that maps F to $\mathbb{R}^3$, and some vector b_t^F in $\mathbb{R}^3$. Being a linear transformation that preserves the metrics (and therefore the inner product) and preserves the orientation, A_t^F must be the restriction of a real proper (i.e. $\det = 1$) orthogonal linear operator O_t^F on $\mathbb{R}^3$ (i.e. $A_t^F = O_t^F|_F$). Recall from linear algebra that the derivative of a differentiable one parameter family of real proper orthogonal linear transformations is a skew-symmetric linear transformation. Thus if f_t on F is differentiable with respect to t, then $S^F = \frac{dA_t^F}{dt}|_{t=0}$ is the restriction of a skew-symmetric linear transformation on F, and $b^F = \frac{db_t^F}{dt}|_{t=0}$ is simply a vector in $\mathbb{R}^3$.

Now we are ready to find a necessary and sufficient condition for a vector field on a plane to be a bending field.

Lemma 2.4. *A vector field B on a plane F is a flexing field if and only if the field vector $B(q)$ at every point q on F is equal to $Sq + b$, where b is a vector in $\mathbb{R}^3$ and S is the restriction of a skew-symmetric linear transformation on F. (In other words, $Su \cdot v = -u \cdot Sv$ for all $u, v \in F$.)*

The necessity part of the lemma was established in the discussion above. Sufficiency is rather immediate as well: a skew-symmetric matrix S is the derivative at $t = 0$ of the following function from $[0, 1)$ to the set of real proper orthogonal matrices: $O(t) := e^{tS}$. Thus $Sq + b$ is the derivative of a isometric affine deformation of the plane $f_t(q) = O(t)q + tb$. QED.

An immediate corollary:

Corollary 2.5. *If B is a flexing field of a polyhedron P, then the field on each face of P must satisfies the condition in lemma 2.4, that is for each face F there exists a skew-symmetric linear transformation S_F from F to $\mathbb{R}^3$ and a vector $b_F \in \mathbb{R}^3$ such that $S_F q + b_F = B(q)$ for all $q \in F$.*

Notice that a linear vector field on a plane is totally defined by the field vector at three non-collinear points on the plane (simply because a plane is totally defined by three non-collinear points on it). In fact, we have the following easy result:

Lemma 2.6. *Let B be a linear vector field on a plane P, and a, b, c are three non-collinear points on P. Choose an affine coordinate system so that the origin is at a, the positive x-axis passes through b, the positive y-axis passes through c, and the z-axis is perpendicular to P. In that coordinate system, the field vector at a point $(x, y, 0)$ is determined as follows:*

$$B((x, y, 0)) = B(a) + \frac{x}{|a-b|}B(b) + \frac{y}{|a-c|}B(c). \tag{2.1}$$

This lemma is easily verified. We skip the proof here.

So the field vectors at the vertices of a polyhedron P completely determine a piecewise linear vector field on P. Thus, to check whether a piecewise linear vector field is a flexing field, it suffices to restrict our attention to the vector field at the vertices. Indeed we have the following theorem:

Theorem 2.7. *A* linear *vector field B on a plane F is a bending field if and only if for three arbitrary points a, b, c on F, $(B(a)-B(b))\cdot(a-b) = (B(b)-B(c))\cdot(b-c) = (B(c) - B(a)) \cdot (c - a) = 0$.*

Proof. The necessary direction is easy: $B(a) - B(b) = S(a) - S(b) = S(a - b)$, where S is the transformation from lemma 2.4. Since S is skew-symmetric, $S(a - b) \cdot (a - b) = -(a - b) \cdot S(a - b)$, and thus $S(a - b) \cdot (a - b) = 0$. Similarly for two others equalities.

The sufficient direction follows from lemma 2.4 and lemma 2.6. Given $B(a), B(b), B(c)$ we can define the bending field as in equation 2.1. Then we can define $u := B(a)$ and $S((x, y, 0)) := \frac{x}{|a-b|}B(b) + \frac{y}{|a-c|}B(c)$. One can tediously check that S defined above satisfies $Sq \cdot r = -q \cdot Sr$ for all $q = (x, y, 0)$ and $r = (x', y', 0)$ on the plane by expanding the dot products. Thus S is a skew symmetric linear transformation on the plane. By lemma 2.4, $B(q) := S(q) + u$ is a bending field, as desired. □

An immediate corollary:

Corollary 2.8. *If a vector field B on the surface of a* triangulated *polyhedron P is a flexing field then every edge $p_i p_j$ of P is perpendicular to $B(p_i) - B(p_j)$, where $B(p_k)$ is the field vector at the vertex p_k.*

Unfortunately, the converse of corollary 2.8 is not true. For two faces F and F' sharing an edge e, there is no guarantee that the transformation associated the

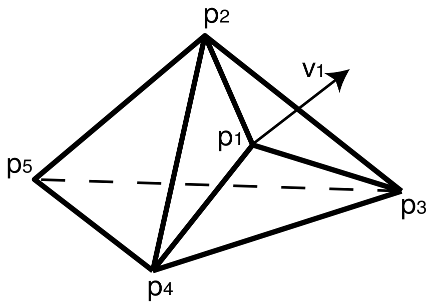

FIGURE 2.2. When $B(p_1) = v_1$ perpendicular to the surface at the flat vertex p_1, and $B(p_2) = \cdots = B(p_5) = 0$. Although these vectors satisfies condition of theorem 2.8, they are not associated with a possible flex.

three vectors at the vertices of F and the transformation associated with the vectors of F' act the same on e. For example consider the field in the situation of a flat vertex as in figure 2.2.

2.3. **Rigidity Matrix.**

Given a polyhedron P, if we introduce an origin, then the vertices p_i will become vectors (though $p_i - p_j$ are independent of the origin) . Then theorem 2.8 says that $(B(p_i) - B(p_j)) \cdot (p_i - p_j) = 0$ for all i, j such that p_i is adjacent to p_j. For simplicity of notation, let us denote $v_i = B(p_i)$ (v stands for velocity). Then it is rather obvious that $v_1, v_2, \ldots, v_n$ satisfies this orthogonal if and only if $v = (v_1, v_2, \ldots, v_n)$ is in the kernel of a matrix $R(P)$ that we call the *rigidity matrix* of P, defined as follows:

Definition 2.9. The columns of the rigidity matrix of P, denoted $R(P)$, are organized into n sets of 3 columns, each of which sets corresponds to a vertex of P. Each row of the rigidity matrix $R(P)$ corresponds to an edge of P. The row corresponding to the edge p_ip_j consists of all zeros, except for the two sets of 3 entries corresponding to p_i and p_j. The entries corresponding to p_i are the coordinates of $p_i - p_j$, and the entries corresponding to p_j are the coordinates of $p_j - p_i$.

For example, if P is a pyramid with vertex p_1 on top of base $p_2p_3p_4p_5$, then $R(P)$ has 8 rows and 15 columns (organized into 5 sets of 3 columns):

$$\begin{pmatrix}
[p_1 - p_2]^T & [p_2 - p_1]^T & 0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 \\
[p_1 - p_3]^T & 0\ 0\ 0 & [p_3 - p_1]^T & 0\ 0\ 0 & 0\ 0\ 0 \\
[p_1 - p_4]^T & 0\ 0\ 0 & 0\ 0\ 0 & [p_4 - p_1]^T & 0\ 0\ 0 \\
[p_1 - p_5]^T & 0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 & [p_5 - p_1]^T \\
0\ 0\ 0 & [p_2 - p_3]^T & [p_3 - p_2]^T & 0\ 0\ 0 & 0\ 0\ 0 \\
0\ 0\ 0 & 0\ 0\ 0 & [p_3 - p_4]^T & [p_4 - p_3]^T & 0\ 0\ 0 \\
0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 & [p_4 - p_5]^T & [p_5 - p_4]^T \\
0\ 0\ 0 & [p_2 - p_5]^T & 0\ 0\ 0 & 0\ 0\ 0 & [p_5 - p_2]^T
\end{pmatrix}$$

The rigidity matrix of P looks very similar to the incidence matrix of the graph of P (i.e. the graph whose vertices are vertices of P, and whose edges correspond to edges of P). Indeed if we replace each block of nonzero entries in $R(P)$ with an 1, and each block of three zero entries with a 0, we get the incidence matrix. The rigidity matrix is (a half of) the matrix of the differential linear map $df(P) : \mathbb{R}^{3n} \to \mathbb{R}^e$ (with e being the number of edges in P), where f is the *rigidity map*, defined as follows:

$$\begin{aligned} f : \mathbb{R}^{3n} &\to \mathbb{R}^e \\ f((p_1, p_2, \dots, p_n)) &= (\dots, |p_j - p_i|^2, \dots). \end{aligned}$$

(See [7] for elaborations of the rigidity map) Indeed, f depends largely on the combinatorial type of P, but when we restrict ourselves to a combinatorial equivalence class of polyhedra, we can safely write f as a function of $p = (p_1, p_2, \dots, p_n) \in \mathbb{R}^{3n}$.

We rewrite the statement above about $\ker R(P)$ in the following lemma (again, we leave it to the reader to verify this obvious result):

Lemma 2.10. *The n-tuple of vectors $v_1, v_2, \dots, v_n$ attached to the vertices $p_1, p_2, \dots, p_n$ of the triangulated polyhedron P, satisfies $(v_i - v_j) \cdot (p_i - p_j) = 0$ for all edges $p_i p_j$, if and only if $v = (v_1, v_2, \dots, v_n) \in \ker R(P)$.*

Corollary 2.11. *If the n-tuple of vectors $v_1, v_2, \dots, v_n$, attached to the vertices $p_1, p_2, \dots, p_n$ of the triangulated polyhedron P, spans a flexing field of P then $v = (v_1, v_2, \dots, v_n) \in \ker R(P)$.*

3. Rigidity of Polyhedra

Now equipped with bending fields and the rigidity matrix, we are ready to attack the rigidity problem of polyhedra. But first of all, let us take a look back at history and see how Cauchy first proved the famous theorem that was later named after him.

3.1. **Cauchy's Rigidity Theorem.**

Theorem 3.1. *Two convex polyhedra composed of pairs of congruent faces, assembled in the same way (i.e. two combinatorially equivalent and isometric polyhedra) are indeed congruent.*

This result immediately implies that convex polyhedra are rigid. Let us give a quick review of Cauchy's brilliant proof.

Sketch of Proof. It suffices to show that the corresponding dihedral angles are equal. Suppose they are not. Then for each edge of P at which the dihedral angle is larger the corresponding dihedral angle of P' with a plus (+). Similarly, mark each edge of P at which the dihedral angle is smaller than the corresponding angle in P'. If the two angles are equal, we mark the edge with a zero. The index of a vertex is the number of sign changes (from + to - or vice versa) when one goes around the outgoing edges at a vertex once.

Lemma 3.2 (Geometric)**.** *No vertex of P has index 2. If a vertex has index 0, then all the outgoing edges are marked with 0s.*

Lemma 3.3 (Topology)**.** *A spherical/planar graph satisfying lemma 1 has all edges marked with 0s.*

The first lemma is sometimes called the Arm Lemma, since Cauchy's argument involves comparing two convex polygonal paths with corresponding sides equal. (Cauchy's first proof of this lemma had a small pitfall that went undetected for decades, but it was finally fixed). The proof of the second lemma is a simple argument using the Euler characteristic of a graph on the sphere. These two lemmas immediately imply all the corresponding angles are equal. □

Cauchy's simple and beautiful proof of a long standing problem has inspired a large literature on the subject of rigidity. The reader can refer to [1] or [7] for elaboration of Cauchy's proof.

3.2. **Infinitesimal Rigidity.** Recall that a polyhedron is rigid if every possible flex is an euclidian motion. Thus rigidity is implied if every possible flexing field is associated with an euclidian motion. The Lie group of all euclidian motions has dimension 6 (3 degrees of freedom for shifting, and 3 degrees of freedom for rotation (rotation around x,y or z axis)). Hence if the dimension of the group of all possible flexing fields is 6 then the polyhedron is rigid. By lemma 2.10, it follows that in this direction, it suffices to show $\dim\ker R(P) = 6$.

Notice that the fact every n-tuple of vectors $v_1, v_2, \ldots, v_n$, satisfying $(v_i - v_j) \cdot (p_i - p_j) = 0$ for all edge $p_i p_j$, is associated with an euclidian motion (i.e. for all i, $v_i = r \times p_i + u$ for some vectors r and u) is stronger than the fact every flexing field is associated with an euclidian motion. This is because not every such n-tuple spans a flexing field. (For the converse of lemma 2.8 is not necessarily true.) Indeed, a polyhedron is said to be *infinitesimally rigid* if every such n-tuple of vectors is associated with an euclidian motion. ([10], [7])

Definition 3.4. A polyhedron P is said to be *infinitesimally rigid* if for every n-tuple of vectors $v_1, v_2, \ldots, v_n$, satisfying $(v_i - v_j) \cdot (p_i - p_j) = 0$ for all edge $p_i p_j$, there exists vectors u and r such that

$$v_i = r \times p_i + u,$$

for all $i = 1, 2, \ldots, n$.

As discussed above, infinitesimal rigidity is stronger than rigidity. (One can also prove this fact without knowing about bending fields. See, for example, [10].) The polyhedron in figure 2.2 is an example of a not infinitesimally rigid but rigid polyhedron. [1]

To make the paper self-contained, we provide a concise version of Aleksandrov's proof (from [3]) of the following infinitesimal rigidity theorem, also known as Dehn's theorem:

Theorem 3.5 (Dehn, 1916)**.** *A strictly convex triangulated polyhedron in three-space is infinitesimally rigid.*

Proof. As discussed previously, it suffices to show that $\dim\ker R(P) = 6$, where $R = R(P)$ is the rigidity matrix of P. Since P is convex, P is homeomorphic to a sphere, and thus $e = 3n - 6$, where e is the number of edges and n is the number of vertices. Recall that R has $3n$ columns and e rows. So $\dim\ker R = 6$ if and only if $\dim\ker R^T = 0$.

[1]The authors do not know an example of a non-degenerate polyhedron (i.e. no flat vertices) that is rigid but not infinitesimally rigid.

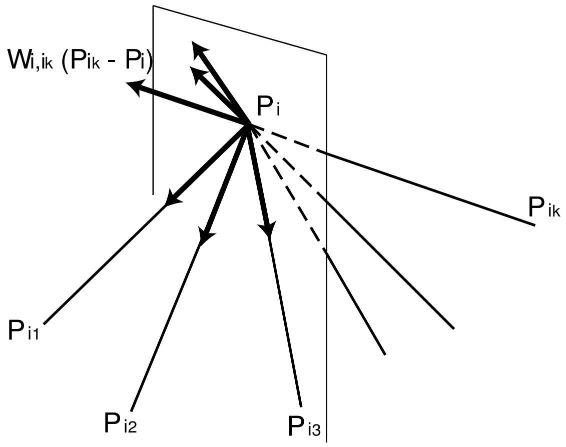

FIGURE 3.1. The separating plane.

Suppose w is a vector in $\mathbb{R}^e$ such that $R^T w = 0$. We shall establish that $w = 0$. We will use an argument similar to that in Cauchy's geometric and topology lemmas. Each edge $p_i p_j$ is assigned a real number $w_{i,j}$ (one of the coordinates of w). The index of a vertex is the number of sign changes (from strict negative to strict positive or vice versa) as one goes around the outgoing edges at that vertex once.

Since $R^T w = 0$, it follows that at an arbitrary vertex p_i, if we cyclically label the vertices adjacent to p_i by $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$, then $\sum_{j=1}^{k} w_{i,i_j}(p_{i_j} - p_i) = 0$. Thus all the w_{i,i_j} cannot be all positive or all negative.

On the other hand, if the index of p_i is 2, then by strict convexity of the polyhedron at p_i, one can separate the positive w_{i,i_j} and the negative w_{i,i_j} by a plane through p_i. (See figure 3.2.) Without loss of generality, suppose $w_{i,i_j} \geq 0$ for $j = 1, \ldots, s$ and $w_{i,i_j} \leq 0$ for $j = s+1, \ldots, k$. Then $\sum_{j=1}^{s} w_{i,i_j}(p_{i_j} - p_i) + \sum_{j=s+1}^{k} w_{i,i_j}(p_{i_j} - p_i) = 0$. Notice that, because of the signs of w_{i_j}, for all $j = s+1, \ldots, k$, $w_{i,i_j}(p_{i_j} - p_i)$ is in the half plane containing $w_{i,i_1}(p_{i_1} - p_i), \ldots, w_{i,i_s}(p_{i_s} - p_i)$ (if we place the heads of all the vectors at p_i). Thus in order for the sum $\sum_{j=1}^{s} w_{i,i_j}(p_{i_j} - p_i) + \sum_{j=s+1}^{k} w_{i,i_j}(p_{i_j} - p_i) = 0$ to be the zero vector, every term must be the zero vector itself. So $w_{i,i_j} = 0$ for all $j = 1, \ldots, k$, and thus the index of p_i is 0, contradictory.

Consequently, we now have a polyhedron homeomorphic to a sphere in which the index of a vertex is either 0 or at least 4, and in the case the index is 0, every outgoing edge at that vertex must be assigned a zero. By Cauchy's topology lemma, every edge must be assigned a zero. This means $w_{i,j} = 0$ for all edges $p_i p_j$, and thus $w = 0$, as desired. □

3.3. **Almost all Polyhedra are Rigid.** In this section, we will establish that in the set of polyhedra homeomorphic to the two-sphere, the set of flexible polyhedra has measure zero. We independently derived the argument that was previously mentioned in H. Gluck's paper [10].[2]

[2]In [10], Gluck stated the set of rigid polyhedra is open and dense in the set of all polyhedra in a given combinatorial type. This statement does not imply that the set of flexible polyhedra

First, let us have the following definition once and for all:

Definition 3.6. Two polyhedra are said to be *combinatorially equivalent* if their graphs are the same. (Recall that the graph of a polyhedron can be obtained from the polyhedron by removing all the interior of the faces.)

This combinatorial equivalence is an equivalence relation. An equivalent class of this relation is called a combinatorial type.

Theorem 3.7. *Let* $[\mathcal{P}]$ *be the set of all triangulated polyhedra homeomorphic to a sphere of a given combinatorial type. Let* $[\mathcal{P}_f]$ *be the set of flexible polyhedra in* $[\mathcal{P}]$*. Then* $[\mathcal{P}_f]$ *has measure zero in* $[\mathcal{P}]$*.*

Proof. As we discussed earlier, a polyhedron is rigid if the dimension of the kernel of its rigidity matrix is 6. Thus it suffices to show that the set of polyhedron P in $[\mathcal{P}]$ such that $\dim\ker R(P) > 6$ has measure zero.

Let us first 'crop' the rigidity matrix so that it becomes a square matrix. To do so, simply remove the nine columns corresponding to the coordinates of the three vertices of a chosen face. Also remove the three rows corresponding to this face's three edges. (One can think of this process as fixing a face under flexing, and thus getting rid of all the euclidian motions.) Then we now have a square $(3n-9)\times(3n-9)$ sub-matrix $\tilde{R}(P)$ of $R(P)$. A polyhedron is rigid if $\dim\ker\tilde{R}(P) = 0$, that is if $\det\tilde{R}(P) = 0$. We shall show a statement bolder than the theorem: the set $\{P \in [\mathcal{P}] | \det\tilde{R}(P) = 0\}$ has measure zero in $[\mathcal{P}]$.

Label the vertices of P so that the face removed from the matrix is $p_{n-2}p_{n-1}p_n$. In the given combinatorial type, the determinant of $\tilde{R}(P)$ is $f(p_1, p_2, \ldots, p_{n-3})$, a polynomial of the three-vectors $p_1, p_2, \ldots, p_{n-3}$.

This polynomial is non-zero. This is because there always exists a convex polyhedron Q in $[\mathcal{P}]$ (draw the representing graph of $[\mathcal{P}]$ on a sphere, and replace the graph's vertices with a polyhedron's vertices and replace the graph's edges with a polyhedron's edges). But we already know from Cauchy's rigidity theorem such a convex polyhedron is rigid. Therefore $\dim\ker R(Q) = 6$, so $\dim\ker\tilde{R}(Q) = 0$, and hence $\det\tilde{R}(Q) \neq 0$. This means $f(q_1, q_2, \ldots, q_{n-3}) \neq 0$.

Now, since f is a non-trivial polynomial, the set of zeros of f has measure zero in the domain of f. Thus the set of flexible polyhedra in $[\mathcal{P}]$ has measure zero in $[\mathcal{P}]$. □

Remark 3.8. If we can show that for polyhedra not homeomorphic to the two-sphere, in every combinatorial type, there exists a polyhedron that is rigid, then the argument above will establish a stronger statement, that almost all polyhedra (regardless of the Euler characteristic) are rigid.

4. The Bellows Conjecture

We now know that though very rare, there are flexible polyhedra. The first flexible polyhedron was found by Connelly in the 70s. After him, people have found a few other examples of fewer faces and vertices. (The polyhedron with the smallest number of vertices and faces so far has 9 vertices and 14 faces. It was counstructed

---

has measure zero; the complement of an open and dense set not necessarily has measure zero. Counter example: Label the rational numbers $q_1, q_2, \ldots$, and choose a positive convergent series $\sum_{i=1}^{\infty} l_i$; the union of the intervals $(q_i - l_i/2, q_i + l_i/2)$ is open and dense in $\mathbb{R}$, but their measure is $\sum l_i < \infty$.

by Klaus Steffen.) The curious reader should refer to [8] for constructions of such polyhedra. It turns out that there are invariants of a flexible polyhedron under flexing, and volume is one of them.

We recall that an continuous deformation of a polyhedron is a *flex* if under which every face simply undergoes a deformation consisting of a continuous family of affine transformations in three-space. The bellows conjecture claims that flexing does not change the volume of a polyhedron.

Theorem 4.1 (The Bellows Conjecture/Theorem)**.** *The volume of a polyhedron remains constant under a flex.*

This result was conjectured by Dennis Sullivan ([15]), and was proved and then polished by Connelly, Sabitov and Walz in [5] in 1997. The proof holds for any polyhedron (regardless of the Euler characteristic, convexity, etc.). Basically, they construct a polynomial whose coefficients only depend on the combinatorial type and the side lengths of the polyhedron. The volume of the polyhedron must be one of the roots of this polynomial. (This idea was probably inspired by the Heron's formula for the area of a triangle.) Therefore there are only a finite number of possibilities for the volume of a polyhedron give its combinatorial type and its edge lengths. And thus, the volume must remain constant under a continuous flexing.

We want to find an alternative proof that is more intuitive in some way and that might lead us to an analog of the bellows conjecture for the smooth case. Bending field seems to be a natural approach.

It suffices to prove the bellows conjecture for triangulated polyhedra (every face is a triangle), because every polyhedron can be triangulated, and if the bellows conjecture is true for this new triangulated polyhedron, it must be also true for the original polyhedron. Therefore in the rest of the paper we continue to restrict our attention to only triangulated polyhedra.

4.1. **First Claim.**

In general, suppose we have a simply closed smooth surface ϕ in $\mathbb{R}^3$ parameterized by $\phi(\alpha, \beta)$, where (α, β) lies in some domain of parameters. Suppose we have a deforming field $v(\alpha, \beta)$ on ϕ, which is associated with a deformation f_t. Suppose dA is an area form on ϕ, then the rate of change of the volume enclosed by ϕ under such deformation f_t is:

$$\frac{dV}{dt}|_{t=0} = \int_\phi v(\alpha, \beta) \cdot n(\alpha, \beta) dA, \tag{4.1}$$

where $n(\alpha, \beta)$ is the unit outward normal vector at $\phi(\alpha, \beta)$.

In the case of a polyhedron, instead of a general deforming field, we have a flexing field. A flexing field on a polyhedron is spanned by the bending vector at the vertices of the polyhedron, so naturally we should have a formula for the rate of change of volume under a bending in terms of only those bending vectors. In fact, we have the following result:

Theorem 4.2. *Suppose P is a polyhedron with a flexing field B. Let v_i denote the bending vector at the vertex p_i. Then the rate of change of the volume of P under the flexing whose velocity field at time $t = 0$ is B is:*

$$\frac{dV}{dt}|_{t=0} = \frac{1}{3} \sum A_{ijk} \quad n_{ijk} \cdot (v_i + v_j + v_k), \tag{4.2}$$

where the sum runs over all faces $p_ip_jp_k$ oriented so that the unit normal vector $n_{ijk} = \frac{(p_j-p_i)\times(p_k-p_i)}{|p_j-p_i||p_k-p_i|}$ is outward with respect to the polyhedron, and A_{ijk} is the area of the face $p_ip_jp_k$.

Proof. Suppose dA is an area form on the surface of the polyhedron. Then, as in equation (4.1), the rate of change of the volume is:

$$\frac{dV}{dt}|_{t=0} = \int_P v(q) \cdot n(q) dA,$$

where $v(q)$ is the bending vector at point q on (the surface of) P, and $n(q)$ is the unit outward normal vector at q.

On each face F whose vertices are p_i, p_j and p_k (oriented so that the normal is outward), let us introduce a coordinate system as in lemma 2.6, i.e. the origin is at p_i, the positive x-axis is through p_j, the positive y-axis is through p_j, and the positive z-axis is in the direction of the vector $(p_j - p_i) \times (p_k - p_i)$. Suppose the angle between $p_j - p_i$ and $p_k - p_i$ is θ $(0 < \theta < \pi)$ (it is also the angle at the vertex p_i of the triangle $p_ip_jp_k$). Then an area form of the face F is $dA_F = \sin(\theta_F)dxdy$. (We assume the polyhedron is non-vertex-degenerate, that is no three collinear vertices). Therefore:

$$\frac{dV}{dt}|_{t=0} = \sum \left(\int_F v(x, y, 0) \cdot n_F dA_F \right), \tag{4.3}$$

where the sum run over all appropriately oriented faces F of P.

Let us evaluate each integral in the sum on the right hand side. Consider a face F whose vertices are p_i, p_j, p_k (assuming this orientation gives an outward normal vector). Then according to lemma 2.6,

$$v(x, y, 0) = v_i + \frac{x}{|p_j - p_i|} v_j + \frac{y}{|p_k - p_i|} v_k$$

Thus

$$\begin{aligned} \int_F v(x, y, 0) \cdot n_F \ \ dA_F &= v_i \cdot n_F \int_F dA_F \\ &+ \frac{v_j \cdot n_F}{|p_j - p_i|} \sin(\theta) \int_F xdxdy \\ &+ \frac{v_k \cdot n_F}{|p_k - p_i|} \sin(\theta) \int_F ydxdy. \end{aligned} \tag{4.4}$$

The first term on the right hand side sum is simply $v_i \cdot n_F A_F$, where A_F is the area of the face F.

On the other hand, if we denote $a = |p_j - p_i|$ and $b = |p_k - p_i|$, then the area on the plane of F bounded by F is described by the inequalities: $0 \le x, y$ and $x/a + y/b \le 1$. Thus

$$\begin{aligned} \int_F xdxdy &= \int_0^b \int_0^{a(1-y/b)} xdxdy \\ &= \frac{a^2 b}{6}. \end{aligned}$$

Therefore the second term on the right hand side of equation (4.4) is $\frac{1}{a}(v_j - v_i) \cdot n_F \sin(\theta) \frac{1}{6} a^2 b = \frac{1}{6} ab \sin(\theta)(v_j - v_i) \cdot n_F = \frac{1}{3} A_F (v_j - v_i) \cdot n_F$. Similarly, the third term is $\frac{1}{3}A_F(v_k - v_i) \cdot n_F$. Summing up all these yields:

$$\begin{aligned}\int_F v(x,y,0)\cdot n_F \quad dA_F &= A_F v_i \cdot n_F + \frac{1}{3}A_F(v_j - v_i)\cdot n_F + \frac{1}{3}A_F(v_k - v_i)\cdot n_F \\ &= \frac{1}{3}A_F n_F \cdot (v_i + v_j + v_k).\end{aligned}$$

Summing up this result for all faces F, we have the desired equality.

□

So the bellows conjecture is a consequence of the following claim:

Claim 4.3. *Suppose P is a polyhedron with n vertices $p_1, p_2, \ldots, p_n$. If $v = (v_1, v_2, \ldots, v_n)$ is any n-tuple of vectors in $\mathbb{R}^3$ satisfying: every edge p_ip_j is perpendicular to $v_i - v_j$, then*

$$\sum A_{ijk} \quad n_{ijk} \cdot (v_i + v_j + v_k) = 0, \tag{4.5}$$

where the sum runs over all faces $p_ip_jp_k$ of P, and $p_ip_jp_k$ is oriented so that $n_{ijk} = \frac{(p_j - p_i)\times(p_k - p_i)}{|p_j - p_i||p_k - p_i|}$ is an outward normal.[3]

Let us check equality (4.5) for some simple cases.

Example 4.4. First of all, let us consider $v_1 = v_2 = \cdots = v_n$ (i.e. the bending vectors $v_1, v_2, \ldots, v_n$ correspond to a shift in $\mathbb{R}^3$). Then equality (4.5) becomes

$$\left(\sum A_{ijk} \quad n_{ijk}\right) \cdot v_1 = 0. \tag{4.6}$$

This equality immediately follows the elementary result that sum of all the Minkowski vectors of a polyhedron is 0:

Lemma 4.5.

$$\sum A_{ijk} \quad n_{ijk} = 0. \tag{4.7}$$

Physically, this theorem says that the total forces of a compressed, uniform gas in a polyhedron on the faces is zero.

Proof. Suppose the face $p_ip_jp_k$ is oriented appropriately. Then

$$\begin{aligned} A_{ijk} \quad n_{ijk} &= \frac{1}{2}(p_j - p_i)\times(p_k - p_i) \\ &= \frac{1}{2}(p_j \times p_k - p_i \times p_k - p_j \times p_i + p_i \times p_i) \\ \Rightarrow A_{ijk} \quad n_{ijk} &= \frac{1}{2}(p_i \times p_j + p_j \times p_k + p_k \times p_i). \end{aligned} \tag{4.8}$$

If we sum this equation up over all faces, then all the terms will cancel each other out. This is because each edge p_ip_j belongs to two adjacent faces, and due to orientation, in one face the cross product associated with this edge is $p_i \times p_j$, while in the other face the cross product is $p_j \times p_i$. □

[3] The vector An is sometimes called the Minkowski vector at a face.

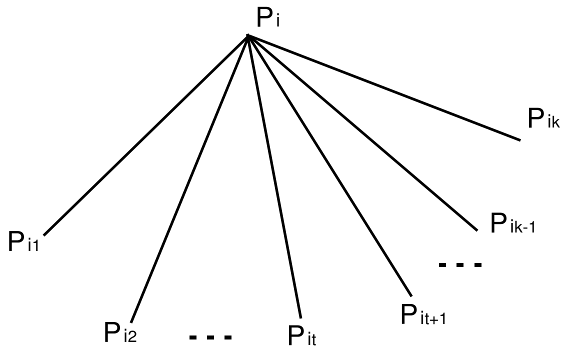

FIGURE 4.1. In a neighborhood of vertex p_i.

Example 4.6. Now, let us look at another case: when $v_i = r \times p_i$ for some vector r and for all $i = 1, 2, \ldots, n$. (This case corresponds to a rotation of the polyhedron around the axis through r.) The proof goes as follows:

$$\begin{aligned} & \sum A_{ijk} \quad n_{ijk} \cdot ((p_i + p_j + p_k) \times r) \\ = & \sum (A_{ijk} \quad n_{ijk} \times (p_i + p_j + p_k)) \cdot r \end{aligned}$$

Now substitute $A_{ijk} n_{ijk}$ with the right hand side of equation (4.8). Then

$$\begin{aligned} & \sum (A_{ijk} \quad n_{ijk} \times (p_i + p_j + p_k)) \cdot r \\ = & \left(\sum \frac{1}{2} (p_i \times p_j + p_j \times p_k + p_k \times p_i) \times (p_i + p_j + p_k) \right) \cdot r \\ = & \frac{1}{2} \left(\sum (p_i \times (p_j \times p_k) + p_j \times (p_k \times p_i) + p_k \times (p_i \times p_j)) \right) \cdot r \\ = & \frac{1}{2} 0 \cdot r. \end{aligned}$$

The last equality is due to the Jacobi identity: $a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$ for all $a, b, c \in \mathbb{R}^3$. So claim 4.3 also holds for this case.

4.2. **Second Claim.**

Claim 4.3 is in a pretty nice form and is rather intuitive (the left hand side of equation 4.5 is the rate of change of the volume if we flex along the field spanned by $v_1, v_2, \ldots, v_n$). However it involves the bending vectors, which are arbitrary in some sense. Fortunately, we can derive another equivalence statement independent of the bending field. This is basically similar to taking the dual of claim 4.3.

First of all, let us rearrange the terms of the sum on the left hand side of equation 4.5. Suppose vertex p_i is adjacent to the vertices $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$, and these k vertices are cyclically oriented so that the right hand rule of the cross products gives us outward normal vectors (see figure 4.2). Then

$$\sum A_{ijk} \quad n_{ijk} \cdot (v_i + v_j + v_k) = \sum_{i=1}^{n} \left(\sum_{t=1}^{k} A_{i i_t i_{t+1}} \quad n_{i i_t i_{t+1}} \right) \cdot v_i. \tag{4.9}$$

(As usual, we understand $i_{l+1} := i_1$.) Let

$$s_i := \sum_{t=1}^{k} A_{ii_ti_{t+1}} \quad n_{ii_ti_{t+1}}, \tag{4.10}$$

and let

$$s := (s_1, s_2, \dots, s_n) \in \mathbb{R}^{3n}. \tag{4.11}$$

Then the left hand side of equation 4.9 is $s \cdot v$, and consequently, equation (4.5) is equivalent to:

$$s \cdot v = 0. \tag{4.12}$$

So claim 4.3 is equivalent to the fact that that s is perpendicular to all spanning bending vectors v. Recall from lemma 2.10 that $v = (v_1, v_2, \dots, v_n)$ satisfies $(v_i - v_j) \cdot (p_i - p_j) = 0$ for all edges p_ip_j of P if and only if v is in the kernel of the rigidity matrix $R(P)$. Thus theorem 4.3 is equivalent to the fact that such a vector s defined above is in the perpendicular vector field to $\ker R(P)$, which is usually denoted $(\ker R(P))^{\perp}$. But we know $(\ker R(P))^{\perp} = \text{span } (R(P))^* = \text{span } R^T(P)$.

Claim 4.7. *Given a polyhedron P, define $s \in \mathbb{R}^{3n}$ as in (4.11) and (4.10). Then there exists a vector $x \in \mathbb{R}^e$ (e is the number of edges in P) such that $R(P)^T x = s$.*

Let us try to study the linear system of equation $R^T(P)x = s$. Let $x = (\dots, x_{i,j}, \dots)$, where the pair i, j corresponds to edge p_ip_j. Then for each vertex p_i of P, using the same notation $i_1, i_2, \dots, i_k$ as in figure 4.2, the subsystem of 3 linear equations corresponding to p_i is:

$$x_{i,i_1}(\mathbf{p}_i - \mathbf{p}_{i_1}) + x_{i,i_2}(\mathbf{p}_i - \mathbf{p}_{i_2}) + \cdots + x_{i,i_k}(\mathbf{p}_i - \mathbf{p}_{i_k}) = \mathbf{s}_i. \tag{4.13}$$

(Here we have used bold letters to help distinguishing between unknown scalars and vector parameters.) This linear system has three equations (corresponding to the three coordinates of the $\mathbf{p}_j$ in $\mathbb{R}^3$) and k unknowns, where k equals the degree of vertex p_i. So there are always at least 3 unknowns.

Furthermore, notice that each unknown $x_{i,j}$ appears in exactly two subsystems: the one corresponding to p_i and the other one corresponding to p_j.

Finally, there is an alternate formula for s_i.

Lemma 4.8. *For an arbitrary vertex p_i, define s_i as in equation (4.10), then*

$$s_i = \frac{1}{2}(p_{i_1} - p_i) \times (p_{i_2} - p_i) + (p_{i_2} - p_i) \times (p_{i_3} - p_i) + \cdots + (p_{i_k} - p_i) \times (p_{i_1} - p_i). \tag{4.14}$$

And thus also:

$$s_i = \frac{1}{2}(p_{i_1} \times p_{i_2} + p_{i_2} \times p_{i_3} + \cdots + p_{i_k} \times p_{i_1}) \tag{4.15}$$

(regardless of the choice of origin).

Proof. The first equation is obvious, since $A_{ii_ti_{t+1}} n_{ii_ti_{t+1}} = \frac{1}{2}(p_{i_t} - p_i) \times (p_{i_{t+1}} - p_i)$. The second equation immediately follows from expanding the first equation. It can also be derived from equation (4.8). □

5. Future Work

The highest priority is of course given to proving either one of claim 4.3 or 4.7, then the bellows conjecture will immediately follow. The authors personally think the first claim appears to be elementary, and the linear system in the second claim is not too complicated and seems promising. We look forward to having a simple and natural proof of the bellows conjecture in this direction.

In a different direction of studying flexible polyhedra, Connelly (in [8]) conjectured that the solid enclosed by a flexible polyhedron always stays equidecomposable to itself under a flex. Two three-dimensional polyhedra P and P' in $\mathbb{R}^3$ are say to be equidecomposable if we can dissect P into a finite number of polyhedral pieces, $P_1, P_2, \ldots, P_k$, so that if two pieces intersect they only intersect on their boundaries (i.e. $P_1 \cup \cdots \cup P_k = P$ and $P_i \cap P_j \subset$ (boundary P_i) $\cup$ (boundary P_j) for all $i \neq j$), and then reassemble them to get P'. As an answer to Hilbert's third problem, Dehn showed in 1902 that two polyhedra are equidecomposable if and only if they have the same volume and the same Dehn invariant. (See [6].)

The Dehn invariant of a polyhedron P is defined to be:

$$D(P) = \sum_e l(e) \otimes_{\mathbb{Q}} a(e), \tag{5.1}$$

where $l(e)$ is the length of edge e, $a(e)$ is the dihedral angle at edge e, and the sum runs over all edges of P.

We know the volume does not change, so all is left to do is to show the Dehn invariant does not change either. It is worthy to note that if we replace the binary operator $\otimes_{\mathbb{Q}}$ in the definition above with the usual multiplication, we have the total mean curvature of the polyhedron P. It has been shown with a very pretty and elementary proof in [4] that the total mean curvature of a flexible polyhedron is invariant under flexing.

References

[1] Aigner and Ziegler, *Proofs from the Book*, 3rd ed, Springer 2004

[2] A. D. Aleksandrov, *On infinitesimal bendings of nonregular surfaces*, Matematicheskiĭ Sbornik, vol. 1 (43), No. 3, 307-321 (1936); English translation: A.D. Alexandrov selected works, Part 1, Gordon and Breach Publishers, Amsterdam 1996.

[3] A. D. Aleksandrov, *Konvexe Polyeder*, Akademie-Verlag, Berlin 1958.

[4] R. Alexander, *Lipschitzian Mappings and Total Mean Curvature of Polyhedral Surfaces, I*, Transactions of the Amer. Math. Society, Vol. 288, No. 2, 1985.

[5] R. Connelly, I. Sabitov, A. Walz, *The bellows conjecture*, Beiträ zur Algebra und Geometrie, Contributions to Algebra and Geometry, Vol. 38 (1997), No. 1, 1-10.

[6] V. Boltianskii, *Hilbert's Third Problem*, 1978.

[7] R. Connelly, *Rigidity*, Chapter 1.7 (pp. 223-271) of the Handbook of Convex Geometry, P.M. Gruber and J.M. Wills (editors), Elsevier, 1993.

[8] R. Connelly: *The Rigidity of Polyhedral Surfaces*, Math. Mag., Vol. 52, No. 5, Nov 1979.

[9] P. Cromwell, *Polyhedra*, Cambridge University Press 1997.

[10] H. Gluck: *Almost all Simply Connected Closed Surfaces are Rigid*, Lecture Notes in Math., 438, Geometric Topology, Springer-Verlag (1975) 225-239.

[11] N. J. Hicks, *Notes on differential geometry*, Van Nostrand Math. Studies 3 (1964).

[12] A. V. Pogorelov, *Extrinsic geometry of convex surfaces*, Transl. Math. Monographs, 35 Amer. Math. Soc. (1973) Chap. IV.

[13] Velimirović: *Analysis of bending of surfaces using program package Mathematica*, Facta Universitatis, Architecture and Civil Engineering, Vol. 2, No. 1, 1999, pp. 15-21.

[14] E. W. Weisstein. *Catenoid.* From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Catenoid.html

[15] E. W. Weisstein. *Bellows Conjecture*. From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/BellowsConjecture.html

Bucknell University, Lewisburg, PA 17837
E-mail address: tphan@bucknell.edu

Pennsylvania State University, University Park, PA 16802
E-mail address: ccb147@psu.edu