

Bicycle Curves

Van Cyr

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Abstract

In this paper we examine bicycle curves from the point of view of Fourier series. We analyze necessary and sufficient conditions for a circle to admit an infinitesimal deformation as a bicycle curve with rotation number ρ and derive conditions equivalent to those obtained by Tabachnikov. We then show that whenever $\rho \in \mathbb{Q}$ is not $\frac{1}{2}$, that there is no such deformation.

Introduction

Two closed curves $\Gamma(t)$ and $\gamma(t)$ in \mathbb{R}^2 are called bicycle curves if, under some parameterization, the distance from $\Gamma(t)$ and $\gamma(t)$ remains constant as t varies. These curves are called ambiguous bicycle curves if there is both a positively and negatively oriented direction in which the two can be traced such that the distance between them remains constant.

A result due to Tabachnikov established that given $\Gamma(t)$, there exists $\gamma(t)$ for which the pair is ambiguous if and only if the arc length parameterization of Γ has the property that for some $0 < \rho < 1$, $|\Gamma(t+\rho) - \Gamma(t)|$ is constant for all t ; in this case $\Gamma(t)$ is called a bicycle curve with rotation number ρ .

Trivially, it is clear that a circle is an example of a bicycle curve with arbitrary rotation number. Interest in this paper, therefore, lies in finding non-circular bicycle curves. As these turn out not to be obvious, it is interesting to determine when a circle can be infinitesimally deformed into a bicycle curve with a given rotation number. This was originally examined by Tabachnikov, who showed that a circle admits an infinitesimal deformation as a bicycle curve with rotation number ρ if and only if $\exists m \in \mathbb{Z}_{\geq 2}$ such that $m \tan(\pi\rho) = \tan(m\pi\rho)$.

Bicycle Curves as Fourier Series

Let $\Gamma(t)$ be a bicycle curve, then it satisfies:

$$|\Gamma'(t)| = 1, \forall t \tag{1}$$

and has the bicycle property

$$|\Gamma(t + \rho) - \Gamma(t)| = \text{const}, \forall t \quad (2)$$

Therefore we will write $\Gamma(t)$ as the Fourier series, with total arc length 1,

$$\Gamma(t) = \sum_{n \in \mathbb{Z}} A_n e^{2\pi i n t}$$

and recognizing that $|z|^2 = z\bar{z}$ for $z \in \mathbb{C}$, (1) and (2) become

$$\begin{aligned} |\Gamma'(t)| &= \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} A_k \overline{A_{k-n}} k(k-n) e^{2\pi i n t} = 1 \\ &\Rightarrow \sum_{k \in \mathbb{Z}} A_k \overline{A_{k-n}} k(k-n) = 0, \forall n \neq 0 \end{aligned} \quad (3)$$

$$\begin{aligned} |\Gamma(t + \rho) - \Gamma(t)| &= \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} A_k \overline{A_{k-n}} (e^{2\pi i k \rho} - 1)(e^{2\pi i(n-k)\rho} - 1) e^{2\pi i n t} = \text{const} \\ &\Rightarrow \sum_{k \in \mathbb{Z}} A_k \overline{A_{k-n}} (\cos(2\pi \rho(\frac{n}{2} - k)) - \cos(2\pi \rho \frac{n}{2})) = 0, \forall n \neq 0 \end{aligned} \quad (4)$$

Conditions for Solutions Close to the Circle

We would like to find sequences (A_n) that satisfy (3) and (4). We will do this in the context of l_2 , the space of infinite sequences (A_n) for which $\sum_n |A_n|^2$ converges. We are interested in approximating the functions given as the left hand sides of (3) and (4) at points near the circle, which corresponds to $A_j = 0$, $\forall j \neq 1$ and $A_1 = 1$, in l_2 .

Therefore we will calculate (3) and (4) at the point

$$\begin{aligned} x(t) &= (\dots, th_{-2}, th_{-1}, th_0, 1 + th_1, th_2, \dots) \\ &= (\dots, H_{-2}, H_{-1}, H_0, H_1, H_2, \dots) \end{aligned}$$

and look at vectors h for which the term linear in t vanishes. This will be exactly when h is tangent to the solution surfaces of (3) and (4).

For a given value of n , (4) gives us

$$\sum_{k \in \mathbb{Z}} H_k \overline{H_{k-n}} (\cos(2\pi \rho(\frac{n}{2} - k)) - \cos(2\pi \rho \frac{n}{2})) = 0$$

has only two terms linear in t , namely those involving H_1 . Therefore we have

$$t(\cos(2\pi \rho(\frac{n}{2} - 1)) - \cos(2\pi \rho \frac{n}{2})) \overline{h_{1-n}} + t(\cos(2\pi \rho(-\frac{n}{2} - 1)) - \cos(2\pi \rho \frac{n}{2})) h_{1+n} = 0$$

which is equivalent to, setting $m=n+1$,

$$h_m = \frac{\cos(2\pi\rho(\frac{m}{2} - \frac{3}{2})) - \cos(2\pi\rho(\frac{m}{2} - \frac{1}{2}))}{\cos(2\pi\rho(\frac{m}{2} + \frac{1}{2})) - \cos(2\pi\rho(\frac{m}{2} - \frac{1}{2}))} \overline{h_{2-m}} \quad (5)$$

and similarly

$$\sum_{k \in \mathbb{Z}} H_k \overline{H_{k-n}} k(k-n) = 0$$

is equivalent to

$$h_m = \frac{2-m}{m} \overline{h_{2-m}} \quad (6)$$

so a vector tangent to both surfaces (and therefore to their intersection) has a non-zero value for h_m if and only if

$$\frac{\cos(2\pi\rho(\frac{m}{2} - \frac{3}{2})) - \cos(2\pi\rho(\frac{m}{2} - \frac{1}{2}))}{\cos(2\pi\rho(\frac{m}{2} + \frac{1}{2})) - \cos(2\pi\rho(\frac{m}{2} - \frac{1}{2}))} = \frac{2-m}{m}$$

This is equivalent to

$$m \tan(\pi\rho) = \tan(m\pi\rho), m \in \mathbb{N}_{\geq 2} \quad (7)$$

which is exactly the condition described for infinitesimal deformations of the circle by Tabachnikov.

The Circle is Rigid for $\rho \in \mathbb{Q}_{\neq \frac{1}{2}}$

Since there is a vector tangent to the solution surfaces of both (3) and (4) at the circle only if there is a solution (7), we will say that any ρ for which no m satisfies (7) is rigid as a bicycle curve of rotation number ρ . We will show that this is the case whenever $\rho \in \mathbb{Q}_{\neq \frac{1}{2}}$.

Lemma : Let $\rho \neq \frac{1}{2}$, then $\tan(n\pi\rho) = n \tan(\pi\rho) \Leftrightarrow \frac{n-1}{n+1} = \frac{\sin((n-1)\pi\rho)}{\sin((n+1)\pi\rho)}$.

Now notice that $\sin(a\pi\rho) = \Im(\exp(2\pi i \frac{a\rho}{2}))$. So if $\rho \in \mathbb{Q}$ has denominator k , then $\frac{\sin((n-1)\pi\rho)}{\sin((n+1)\pi\rho)}$ is the quotient of the imaginary parts of $2k^{\text{th}}$ roots of unity.

Lemma : If p is prime then the imaginary parts of the roots of unity of order p are never non-trivial rational multiples of each other

Proof: Let $\omega = \exp(\frac{2\pi i}{p})$, then from field theory we know:

- (1) The dimension of $\mathbb{Q}[\omega]$ as a vector space over \mathbb{Q} is $\phi(p)=p-1$
- (2) By symmetry $\Re(\omega^r) = \Re(\omega^{-r})$ and $\Im(\omega^r) = -\Im(\omega^{-r})$ for every $r \in \mathbb{Z}$
- (3) $1 + \omega + \omega^2 + \dots + \omega^{p-1} = 0$
- (4) All elements of $\mathbb{Q}[\omega]$ are linear combinations of elements of $\{\omega^r \mid 0 \leq r \leq p-1\}$

Therefore consider $\{\Re(\omega), \Re(\omega^2), \dots, \Re(\omega^{\frac{p-1}{2}})\} \cup \{\Im(\omega), \Im(\omega^2), \dots, \Im(\omega^{\frac{p-1}{2}})\}$. By (2) it is clear that we can construct each element of $\{\omega^r \mid 0 < r \leq p-1\}$ and by (3) this is enough to construct the number ω^0 . Since linear combinations of these elements can construct the whole field and there are exactly $p-1$ of them, they must form a basis for $\mathbb{Q}[\omega]$. Finally, notice that this implies that no imaginary part of a primitive p^{th} root of unity is a rational multiple of any other, unless the two differ by a factor of ± 1 . \square

Lemma : If p is prime then the imaginary parts of the roots of unity of order p^r are never non-trivial rational multiples of each other

Proof: We will accomplish this by induction on r .

Assume that the result holds in the field of order p^{r-1} and moreover that in that field, there is a basis of all either purely real or purely imaginary elements for which the set of basis elements used to write ω^s as a linear combination is unique among roots of unity of order $r-1$.

Again we know some facts about the cyclotomic field of order r

- (1) The dimension of $\mathbb{Q}[\omega]$ as a vector space over \mathbb{Q} is $p\phi(p^{r-1})$
- (2) A basis for $X_0 = (1, \omega^p, \omega^{2p}, \dots, \omega^{p(p^{r-1}-1)})$ can be given in the assumed form
- (3) A basis for $X_q = (\omega^q, \omega^{p+q}, \omega^{2p+q}, \dots, \omega^{p(p^{r-1}-1+q)})$ can be generated by multiplying each element of the basis for by ω^q .
- (4) The real and imaginary parts of X_q for $1 \leq q \leq \frac{p-1}{2}$ generate the real and imaginary parts of X_q for $\frac{p-1}{2} < q < 0$

Generate a basis for X_0 as before. Let the first half of a basis for X_q for $1 \leq q \leq \frac{p-1}{2}$ be generated by multiplying each element of the basis for X_0 by $\Re(\omega^q)$ and the other half by multiplying each element by $\Im(\omega^q)$. This set of $p\phi(p^{r-1})$ elements can be used to write every element of $\{\omega^r \mid 0 \leq r \leq p-1\}$ as a linear combination. Further since the elements in X_q are in the assumed form for any q and for distinct q the basis for X_q are independent, this basis for $\mathbb{Q}[\omega]$ also has the above property. Hence the result follows by induction. \square

Lemma : If $\gcd(n,m)=1$ and the imaginary parts of the roots of unity of orders n and m , respectively, are never non-trivial rational multiples of each other, then the same is true for roots of unity of order nm

Proof: Assume that the roots of unity of orders n and m all satisfy the hypothesis of the previous lemma. Notice that any root of unity of order nm can be written as the product of a root of order n and a root of order m . Therefore a basis for the cyclotomic field of order nm is given by taking products of basis elements of the cyclotomic fields of orders n and m , respectively (since

$\gcd(n,m)=1$, $\phi(nm) = \phi(n)\phi(m)$). Therefore in the bases under which the hypothesis holds, the new basis clearly also has the property. Therefore the result follows. \square

Theorem : $\frac{n-1}{n+1} = \frac{\sin((n-1)\pi\rho)}{\sin((n+1)\pi\rho)}$ has no solutions for $n \in \mathbb{N}_{\geq 2}$ $\rho \in \mathbb{Q}$

Assume that $\rho \in \mathbb{Q}$ with denominator k . Observe that $\sin((n-1)\pi\rho) = \Im(e^{(2\pi i \frac{(n-1)\rho}{2})})$ and similarly $\sin((n+1)\pi\rho)$, so their quotient is rational and not equal to ± 1 only if there is a k^{th} root of unity whose imaginary part is a non-trivial rational multiple of the imaginary part of another k^{th} root of unity. Write $k = p_0^{a_0} p_1^{a_1} \cdots p_n^{a_n}$ as the prime decomposition of k and repeatedly apply the previous two lemmas to see that this is not the case. \square

Corollary : $n \tan(\pi\rho) = \tan(n\pi\rho)$ has no solutions for $n \in \mathbb{N}_{\geq 2}$ $\rho \in \mathbb{Q}$

It is easily checked, by considering the representation of tangent as exponentials, that $n \tan(\pi\rho) = \tan(n\pi\rho) \Leftrightarrow \frac{n-1}{n+1} = \frac{\sin((n-1)\pi\rho)}{\sin((n+1)\pi\rho)}$. so it follows that the former has no solutions for $n \in \mathbb{N}_{\geq 2}$ $\rho \in \mathbb{Q}$. \square

References

- [1] S. Tabachnikov. Tire track geometry: variations on a theme. Available at <http://www.math.psu.edu/tabachnikov/papers/preprints.html>.