MINIMAL NON-COMPOSITE RECTANGLE EXCHANGE TRANSFORMATION

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Abstract. We define the property of non-composition for rectangle exchange transformation. Subsequently, we show the existence of a non-composite rectangle exchange transformation that is minimal. Other approaches for more general results for these types of transformations are discussed.

1. Introduction

Suppose the space M is the semi-interval $[0,1)$, $\xi = (\Delta_1, \ldots, \Delta_r)$ is a partition of M into $r \geq 2$ disjoint semi-intervals, numbered from left to right, and let $\pi = (\pi_1, \ldots, \pi_r)$ be a permutation of the numbers $(1, 2, \ldots, r)$.

Definition 1.1. Suppose $T : M \to M$ is a translation $T_{\alpha_i}(x) = x + \alpha_i \pmod 1$ on each of the semi-intervals Δ_i and "exchanges" the semi-intervals according to the permutation π, i.e., the semi-intervals $T(\Delta_i) = T_{\alpha_i}(\Delta_i) = \Delta_i'$ adhere to each other in the order $\Delta'_{\pi_1}, \ldots, \Delta'_{\pi_r}$ then T is said to be the *interval exchange transformation* corresponding to the partition ξ and the permutation π.

Under the identification of $M = [0,1)$ with a circle S^1, the exchange of two segments corresponds to rotations $S^1 \to S^1$. Clearly, the translation

$$f : S^1 \to S^1$$
$$\theta \mapsto \theta + 2\pi\omega$$

can be an interval exchange transformation. If $\omega \notin \mathbb{Q}$, f is aperiodic.

Definition 1.2. A map $f : M \to M$ is *recurrent* if $\forall x \in M$ and for some $\epsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $|x - f^n(x)| < \epsilon$.

Definition 1.3. A map $f : M \to M$ is *minimal* if and only if x is transitive $\forall x \in M$.

Definition 1.4. A bijective map $f : M \to M$ is an *isometry* if and only if the map preserves distance on M.

Here we have a classical result that will be of interest later.

Theorem 1.5. *For $\omega \notin \mathbb{Q}$, the translation*

$$f : S^1 \to S^1$$
$$\theta \mapsto \theta + 2\pi\omega$$

is minimal.

Proof. Fix $\epsilon > 0$ and choose $x_0 \in S^1$. Since S^1 is minimal, it can be covered by a finite number of intervals of length ϵ. $\omega \notin \mathbb{Q}$ implies that f is aperiodic. Since $\forall z, z' \in O(x)$, $z \neq z'$, it follows from the Pigeonhole Principle that $\exists\ x_m, x_n \in O(x_0)$ s.t. $|x_m - x_n| < \epsilon$. We assume $m > n$, w.l.o.g. f is an isometry, so $|x_0 - x_{m-n}| < \epsilon$. $x_{m-n} \in O(x_0)$, and therefore f is recurrent for all ϵ. Then for any x_0, some iteration of f lands at any given distance to x_0, meaning x_0 is transitive and f is minimal. □

Now let M be the unit square $[0,1)$x$[0,1)$. ξ is now a partition of M into r disjoint rectangles. π is defined similarly under some index of the rectangles.

Definition 1.6. A map $T : M \to M$ is a *rectangle exchange transformation* if

$$T_{\alpha_i}(\mathbf{x}) = \mathbf{x} + \alpha_{\mathbf{i}}(mod 1)$$

"exchanges" the subsections according to the permutation π as in Definition 1.1.

Analogously, under identification of the space M with a torus $\mathbb{T}^2$, the exchange of two rectangles corresponds to rotations $\mathbb{T}^2 \to \mathbb{T}^2$. The 1x1 rotation of the torus

$$f : \mathbb{T}^2 \to \mathbb{T}^2$$
$$(x, y) \mapsto (x + 2\pi\omega_1, y + 2\pi\omega_2)$$

is a rectangle exchange map with a suitable partition. If ω_1 or ω_2 are irrational, f will be aperiodic.

Theorem 1.7. *If ω_1 and ω_2 are irrational, f is recurrent.*

Proof. Fix $\epsilon > 0$ and choose $\mathbf{x_0} \in M$. Since M is compact, it can be covered by a finite number of open balls of radius ϵ. If M is covered by n ϵ-balls, the aperiodicity of f and the Pigeonhole Principle guarantee that after at most $n + 1$ iterations, $\exists\ \mathbf{x_{m_1}}, \mathbf{x_{m_2}} \in$O$(\mathbf{x_0})$ s.t. $|\mathbf{x_{m_1}} - \mathbf{x_{m_2}}| < \epsilon$. Since f is an isometry, it follows that $|\mathbf{x_0} - \mathbf{x_{m_2 - m_1}}| < \epsilon$. □

Remark 1.8. This proof does not work for any more complicated torus rotations (such as the 2x1 case that will be seen later), as those maps are not isometries near the discontinuity.

2. Existence of Non-Composite Rectangle Exchange Transformations

Definition 2.1. A rectangle exchange transformation is *composite* if it can be expressed as the composition of two interval exchange transformations. If a rectangle exchange transformation is not composite, it is *non-composite*.

Proposition 2.2. *A rectangle exchange map $T : M \to M$ is non-composite if and only if $\exists(x_0,y_0)$, $(x_1,y_1) \in M$ s.t., w.l.o.g., $y_0 = y_1$ and $T_y(y_0) \neq T_y(y_1)$.*

Proof. Suppose such a set of points exist. Were T to be expressible as the composition of two interval exchange maps, all points along a given line would be mapped to the same line under each interval exchange. But since those points exist s.t. they do not get mapped to the same line, this is not the case, and so T is non-composite.

This criterion is easier to check for than the definition itself, and we make use of it later.

Lemma 2.3. *If f and g are rectangle exchange transformations, $h := g \circ f$ is a rectangle exchange transformation.*

Proof. Let ξ_f be a partition of M in the domain of f, and ξ_g a partition of M in the domain of g s.t. $f(\xi_f) = \xi'_f$ and $g(\xi_g) = \xi'_g$. $\xi_{g\circ f} := \xi'_f \cap \xi_g$ is a partition of M in the range of f and the domain of g. Therefore, we have

$$f(f^{-1}(\xi_{g\circ f})) = \xi_{g\circ f} = g^{-1}(g(\xi_{g\circ f}))$$

and so

$$h(f^{-1}(\xi_{g\circ f})) = g(\xi_{g\circ f})$$

It's easy to see that h exchanges the rectangles according to π_f and then π_g, and so it is a rectangle exchange map.

Theorem 2.4. *The transformation*

$$\begin{gathered} T : M \to M \\ x \mapsto x + \beta \\ y \mapsto \begin{cases} y + \omega_1, & x < a \\ y + \omega_2, & x \geq a \end{cases}, a \in (0,1), \omega_1 \neq \omega_2 \end{gathered}$$

is a non-composite rectangle exchange transformation.

Proof. Consider the following functions:

$$\begin{gathered} f : M \to M \\ x \mapsto x + \beta, \\ g : M \to M \\ y \mapsto \begin{cases} y + \omega_1, & x < a \\ y + \omega_2, & x \geq a \end{cases}, a \in (0,1), \omega_1 \neq \omega_2 \end{gathered}$$

These are both rectangle exchange transformations, as noted earlier, since they are equivalent to rotations on $\mathbb{T}^2$. $T = f \circ g$, and so by Lemma 2.3, T is also a rectangle exchange map. Take the two points $\mathbf{x} = (a - \frac{1}{100}, y)$ and $\mathbf{x}' = (a + \frac{1}{100}, y)$. After applying T, we have:

$$T(\mathbf{x}) = (a - \frac{1}{100} + \beta, y + \omega_1)$$
$$T(\mathbf{x}') = (a + \frac{1}{100} + \beta, y + \omega_2)$$

Since $\omega_1 \neq \omega_2$, the map is non-composite. □

3. Minimality

Here we consider T with $\beta = \frac{1}{2} + \frac{1}{2n_1} + \frac{1}{2n_1 n_2} + \cdots$, $n_k \in \mathbb{N}$, $n_{k+1} > n_k$, $a = \frac{1}{2}$, and $\omega_1, \omega_2 \notin \mathbb{Q}$. β converges since as $k \to \infty$, $\{n_k\} \to 0$ and it is well approximated by rational numbers, it converges to an irrational number. So T will be aperiodic.

The goal of this section is to show that for suitably chosen $\{n_k\}$, T is minimal. This will be done in Theorem 3.8. We introduce two sets:

$$B_k^{\frac{1}{2}} = \{(x,y) \in M | \frac{1}{2} - (\frac{1}{n_{k+1}} + \frac{1}{n_{k+1}n_{k+2}} + \cdots) \leq x < \frac{1}{2}\}$$
$$B_k^1 = \{(x,y) \in M | 1 - (\frac{1}{n_{k+1}} + \frac{1}{n_{k+1}n_{k+2}} + \cdots) \leq x < 1\}$$

Proposition 3.1. *$\forall x \in M \setminus (B_0^{\frac{1}{2}} \cup B_0^1)$, $T^2(x)$ is an isometry.*

Proof. Pick a point $\mathbf{x} = (x,y) \in M \setminus (B_0^{\frac{1}{2}} \cup B_0^1)$. We get that

$$T^2(x) - x = 1 + \frac{1}{n_1} + \frac{1}{n_1 n_2} + \cdots (mod 1) = \frac{1}{n_1} + \frac{1}{n_1 n_2} + \cdots .$$

By supposition, this is a very small translation. Since both T and T^2 move the point to a different section of M, $T^2(y) - y = \omega_1 + \omega_2$ (mod 1). As these distances are independent of x and y, T^2 is an isometry. □

Lemma 3.2. *$\forall x \in M \setminus B_k^{\frac{1}{2}} \cup B_k^1$, $T^{2n_k}(x)$ is an isometry.*

Proof. This follows from Prop. 3.1. For some $(x,y) \in M \setminus B_k^{\frac{1}{2}} \cup B_k^1$,

$$T^{2n_k}(x_0) = x_0 + \frac{1}{n_{k+1}} + \cdots$$

which is extremely small. As in the proof of Prop. 3.1, $T^{2n_k}(y_0) = n_k(\omega_1 + \omega_2)$ (mod 1). □

Remark 3.3. Lemma 3.2 makes it obvious why we require the exclusion of the points in $B_k^{\frac{1}{2}}$ and B_k^1. Consider, w.l.o.g., an $\mathbf{x_0} \in B_k^{\frac{1}{2}}$. $T^2(y_0) - y_0 = 2\omega_1$. Since this differs from the rest of the points in M, T^2, and so T^{2n_k}, is not an isometry $\forall x \in M$.

Proposition 3.4. *$B_k^{\frac{1}{2}} \to \emptyset$ and $B_k^1 \to \emptyset$ as $k \to \infty$.*

Proof. Since $\{n_k\} \to \infty$ as $k \to \infty$, $\{\frac{1}{n_k} + \frac{1}{n_k n_{k+1}} + \cdots\} \to 0$. By definition, card($B_k^1 2$)$\to 0$ and card(B_k^1)$\to 0$ as $k \to \infty$. Therefore, both approach $\emptyset$. □

Remark 3.5. Prop. 3.4 also gives us that diam($B_k^{\frac{1}{2}}$) and diam(B_k^1), and so their cardinalities, both go to 0 as $n_{k+1} \to \infty$.

Definition 3.6. A map $f : M \to M$ is *ϵ-dense* for some ϵ if and only if for any open covering of M with balls of radius ϵ, every orbit contains at least one point in each ball.

Theorem 3.7. *T is $\frac{1}{10^k}$-dense $\forall k$ with suitably chosen $\{n_k\}$.*

Proof. M is compact, so it can be covered by a finite number of balls of radius $\frac{1}{10^k}$. T^2 moves all $\mathbf{x} = (x,y) \in M \setminus B_k^{\frac{1}{2}} \cup B_k^1$ by a very small amount. Thus, we can consider the difference negligible between $\mathbf{x}$ and its image under T^2. By Lemma 3.1 and Theorem 1.5, we know $\exists N \in 2\mathbb{N}$ s.t. $T^N(y) - y < \frac{1}{10^{k+2}}$[1]. Since β (in particular, $N \cdot \beta$) is dependant on the elements of $\{n_m\}$, we can choose n_{k+1} so that $T^N(x) - x < \frac{1}{10^{k+2}}$. This takes care of all of the balls solely in $M \setminus B_k^{\frac{1}{2}} \cup B_k^1$.

[1] This corresponds to $\frac{1}{100} \cdot \frac{1}{10^k}$, which we choose since it's much smaller than the desired radius.

The points in $B_k^{\frac{1}{2}} \cup B_k^1$ are taken care of by adding another requirement on the selection of n_{k+1}. By the observation in Remark 3.5, we know we can control the size of these two "bad" sets by using n_{k+1}. So we choose n_{k+1} so diam$(B_k^{\frac{1}{2}})$ and diam$(B_k^1) < \frac{1}{10^{k+2}}$. This means that for any covering of M, the two "bad" sets are contained well within several balls of radius $\frac{1}{10^k}$, all of which have already been accounted for. So even if we have a point in one of these sets, if we wait for its orbit to exit the set (since β is not affected), we will still hit all of the balls with points in the orbit. □.

Theorem 3.8. *T is minimal on M with suitably chosen $\{n_k\}$.*

Proof. This follows directly from Theorem 3.7. Since the appropriately constructed T is $\frac{1}{10^k}$-dense $\forall k$, every point is transitive, since we can make the points be ϵ-dense for any ϵ we want. □

Remark 3.9. The previous results therefore give us the three conditions we want on $\{n_k\}$ to construct T as a minimal map. They are:
(i) n_{k+1} should be s.t. $T^N(x) - x < \frac{1}{10^{k+2}}$;
(ii) n_{k+1} should be s.t. diam$(B_k^{\frac{1}{2}})$ and diam$(B_k^1) < \frac{1}{10^{k+2}}$;
(iii) n_{k+1} should be large enough so that the results from the previous stage are not undone.

Condition (iii) is not that important in the construction, since it follows from the other two.

Under this construction, T is a minimal non-composite rectangle exchange transformation. However, even for all T in the class brought up in Section 2, this construction produces a small number of βs that work, let alone for all other possible non-composite maps. We think there should exist more maps of this sort, but we did not have time this summer to explore some other approaches we thought had promise for more general results. Section 4 mentions two such approaches oriented around results from related areas.

4. Other Approaches

The first such result we looked at is from Keane [1], concerning interval exchange maps.

Definition 4.1. A permutation π is *irreducible* if for each $1 \leq j \leq n-1$,

$$\pi(\{1, 2, \ldots, j\}) \neq (1, 2, \ldots, j)$$

Definition 4.2. An interval exchange transformation T is *irrational* if π is irreducible and if the only rational relations between the numbers $\alpha_1, \ldots, \alpha_n$ are multiples of $\alpha_1 + \cdots + \alpha_n = 1$.

Definition 4.3. T satisfies the *minimality condition* if and only if:
(i) T is aperiodic;

(ii) if F is a finite union of half-open intervals whose endpoints all belong to the countable set

$$D^{\infty} = \bigcup_{i=0}^{n-1} O(\delta_i) \cup \{1\}$$

where δ_i is the right endpoint of the sub-interval Δ_i, then $TF = F$ implies $F = M$ or $F = \emptyset$.

Theorem 4.4. *T satisfies the minimality condition if and only if $O(x)$ is dense in M for each $x \in M$.*

Theorem 4.5. *If T is irrational, then T satisfies the minimality condition.*

The first problem we ran into in generalizing this result is that we did not notice an obvious natural extension of irreducibility for rectangle exchange transformations where the partition is non-symmetric. There is a good generalization to those rectangle exchange transformations that are composite, but as we are not dealing with those, we do not mention it. Even so, this result is quite powerful in describing which interval exchange maps are minimal, and even a weakened version could provide a large group of minimal rectangle exchange transformations.//.
The second result we found in Katok & Hasselblatt [2]. It gives a criterion for a continuous torus shift to be minimal.

Proposition 4.6. *Consider the torus $\mathbb{T}^2$, a function $\varphi(x) : S^1 \to R$, and a map $f : (x, y) \mapsto (x + \alpha, y + \varphi(x))$ of $\mathbb{T}^2$. Then either $\varphi(x) = \Phi(x + a) - \Phi(x) + r$ for some continuous $\Phi : S^1 \mapsto \mathbb{R}$ and $r \in \mathbb{Q}$ or f is minimal.*

The discontinuity in the general class of 2x1 rotations we looked at in Section 2 is the reason this result needs to be modified. We ran out of time trying to consider how to modify the result so as to not use continuity. This approach seems more promising than that from Keane due to the stronger similarity between torus shifts and non-composite rectangle exchange maps than between interval exchange maps and rectangle exchange maps.

References

1. M. Keane. "Interval Exchange Transformations." *Math. Z.*, **141** (1975), p. 25-31.
2. A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge (1996).

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