

PROPERTIES OF THE ITERATES OF THE WEIERSTRASS- \wp FUNCTION

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This paper is dedicated to our authors.

ABSTRACT. We shall here discuss several properties of the Weierstrass- \wp function, as defined on the fundamental parallelogram \mathbb{C}/Γ , where \mathbb{C} is the complex plane and Γ is the lattice generated by ω_1 and ω_2 . Using the addition formula for $\wp(z_1 + z_2)$, we shall develop a recurrence relation for $\wp(nz)$ in terms of $\wp(z)$. We shall examine the degree of this expression, some coefficients, and patterns concerning the poles of this function. We shall also consider the geometric interpretation of taking an arbitrary z_0 and adding it to itself, both in the fundamental parallelogram \mathbb{C}/Γ and the elliptic curve generated by $\wp(z)$ and $\wp'(z)$.

1. PRELIMINARIES

1.1. Chebyshev Polynomials.

Definition 1.1. The first-kind Chebyshev Polynomial $T_n(x)$ is a polynomial in x of degree n defined by:

$$(1.1) \quad \cos(n\theta) = T_n(x) \text{ where } x = \cos \theta$$

Theorem 1.2. *The fundamental recurrence relation that generates all of the polynomials $T_n(x)$ is as follows:*

$$(1.2) \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \text{ for } n = 2, 3, \dots$$

with initial conditions

$$(1.3) \quad T_0(x) = 1 \text{ and } T_1(x) = x$$

The nice properties of the Chebyshev Polynomials lead us to develop a recurrence relation for $\wp(nz)$ and look at the properties of the resulting explicit rational function.

1.2. Weierstrass- \wp Function.

Definition 1.3. Let points $\omega_1, \omega_2 \in \mathbb{C}$ have non-colinear position vectors. Now define the lattice Γ as all points of the form

$$(1.4) \quad \omega \in \Gamma \iff \omega = a\omega_1 + b\omega_2 \quad \forall a, b \in \mathbb{Z}$$

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The Weierstrass- \wp function is defined as:

$$(1.5) \quad \wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma, \omega \neq 0} \left(\frac{1}{z - \omega} - \frac{1}{\omega^2} \right) \quad \forall z \in \mathbb{C}$$

Remark 1.4. Notice that $\wp(z)$ is even, and more importantly, that $\wp(z)$ is double periodic, with periods of ω_1 and ω_2 , ie

$$(1.6) \quad \wp(z + \omega_1) = \wp(z + \omega_2) = \wp(z) \quad \forall z \in \mathbb{C}$$

Because of this, we may examine $\wp(z)$ on a fundamental parallelogram \mathbb{C}/Γ defined by ω_1 and ω_2 . Notice that $\wp(z)$ contains its only poles at lattice points, and these are order two poles.

Theorem 1.5. *Using power series expansions and pole/zero comparisons, we see that*

$$(1.7) \quad \wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where

$$(1.8) \quad g_2 = 60 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^4} \text{ and } g_3 = 140 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^6}$$

and that

$$(1.9) \quad \wp'(z) = \frac{-2}{z^3} + 6 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^4} z + 20 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^6} z^3$$

Remark 1.6. From here we see that $\wp'(z)$ is also double-periodic on our lattice, and that it is odd. Because of this, we see that the half-periods must be zeroes of $\wp'(z)$. In our fundamental parallelogram, we have:

$$(1.10) \quad \wp' \left(\frac{\omega_2}{2} \right) = \wp' \left(\frac{\omega_1}{2} \right) = \wp' \left(\frac{\omega_1 + \omega_2}{2} \right) = 0.$$

We also see that the points $(\wp(z), \wp'(z))$ lie on the curve $y^2 = 4x^3 - g_2x - g_3$. This is an elliptic curve with roots at $x = \wp \left(\frac{\omega_1}{2} \right), \wp \left(\frac{\omega_2}{2} \right), \wp \left(\frac{\omega_1 + \omega_2}{2} \right)$.

Using the relationship between $\wp(z)$ and this curve, we can now begin to derive a recurrence relation.

2. RECURRENCE RELATION FOR $\wp(nz)$

Theorem 2.1. *For $z_1, z_2 \in \mathbb{C}$,*

$$(2.1) \quad \wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4} \left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)} \right)^2$$

If $z_1 \equiv z_2 \pmod{\Gamma}$, the limit as $z_2 \rightarrow z_1$ gives

$$(2.2) \quad \wp(2z_1) = -2\wp(z_1) + \frac{1}{4} \left(\frac{\wp''(z_1)}{\wp'(z_1)} \right)^2$$

Now we can give a recurrence relation for $\wp(nz)$:

Theorem 2.2.

3. BASIC PROPERTIES OF $\wp(nz)$

Theorem 3.1.

$$(3.1) \quad \wp(nz) = P_n(\wp(z)) = \frac{1}{n^2} + R_n(\wp(z))$$

where R_n is some rational function of $\wp(z)$ with degree less than 0, $\forall n \in \mathbb{Z}$

Proof. We write the power series expansion for $\wp(z)$ at the origin:

$$(3.2) \quad \wp(z) = \frac{1}{z^2} + 3 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^4} z^2 + 5 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^6} z^4 + \dots,$$

so

$$(3.3) \quad \wp(nz) = \frac{1}{n^2 z^2} + 3 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^4} n^2 z^2 + 5 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^6} n^4 z^4 + \dots$$

Define:

$$(3.4) \quad H_n(z) = 3 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^4} n^2 z^2 + 5 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^6} n^4 z^4 + \dots$$

so

$$(3.5) \quad \wp(nz) = \frac{1}{n^2 z^2} + H_n(z)$$

Notice that as $z \rightarrow 0$, $H_n(z) \rightarrow 0$ as well. We also know that $\wp(nz) = P_n(\wp(z))$. Substitute in our expansion:

$$(3.6) \quad \wp(nz) = P_n \left(\frac{1}{z^2} + H_1(z) \right)$$

Since $P_n(x)$ is some rational function of x , define $\tilde{P}_n(x)$ as the quotient with no term of degree less than 0, and $R_n(x)$ as the remainder of degree less than 0, so:

$$(3.7) \quad \begin{aligned} \wp(nz) = P_n \left(\frac{1}{z^2} + H_1(z) \right) &= \tilde{P}_n \left(\frac{1}{z^2} + H_1(z) \right) + R_n \left(\frac{1}{z^2} + H_1(z) \right) \\ &= \frac{1}{n^2 z^2} + H_n(z) \end{aligned}$$

First note that as $z \rightarrow 0$, $(\frac{1}{z^2} + H_1(z)) \rightarrow \infty$. But since $\deg R_n < 0$, $R_n(\frac{1}{z^2} + H_1(z)) \rightarrow 0$, so R_n will not contribute to the $\frac{1}{z^2}$ term. As for the \tilde{P}_n term,

$$(3.8) \quad \tilde{P}_n \left(\frac{1}{z^2} + H_1(z) \right) = a_0 + a_1 \left(\frac{1}{z^2} + H_1(z) \right) + a_2 \left(\frac{1}{z^2} + H_1(z) \right)^2 + \dots$$

Because R_n does not contribute in the $\frac{1}{z^2}$ term, we see that $a_1 = \frac{1}{n^2}$, and because the expression for $\wp(nz)$ contains no constant terms and no terms of degree less than 2, we see that $a_0 = a_2 = a_3 = 0$, so,

$$(3.9) \quad \begin{aligned} P_n(\wp(z)) &= \tilde{P}_n(\wp(z)) + R_n(\wp(z)) \\ &= \frac{1}{n^2} \wp(z) + R_n(\wp(z)) \end{aligned}$$

□

Remark 3.2. From this we see that the rational function $P_n(\wp(z))$ is always of degree 1

Lemma 3.3. z_0 is a pole of $\wp(nz)$ if and only if $nz_0 \in \Gamma$, or $nz_0 = a\omega_1 + b\omega_2$ so $z_0 = \frac{a\omega_1 + b\omega_2}{n}$ for some $a, b \in \mathbb{Z}$

Theorem 3.4. $\wp(nz) = P_n(\wp(z))$ contains n^2 roots

Proof. In the above lemma, clearly both a and b can take any integer value from 0 to $n - 1$, so there are n^2 poles of $\wp(nz)$.

But as $\wp(nz) = P_n(\wp(z))$, then if a point z_0 is a pole of $\wp(nz)$, then $\wp(z_0)$ must be a pole of $P_n(\wp(z))$, which means that $\wp(z_0)$ is a root of the denominator of the rational function $P_n(\wp(z))$. The one exception is the point $z_0 = 0$, because $\wp(0) = \infty$, which is not the root of any polynomial. Since $\wp(z)$ is well defined everywhere but on the lattice points, this means that the denominator of $P_n(\wp(z))$ must have $n^2 - 1$ roots, or must be of degree $n^2 - 1$. We proved earlier that $\deg P_n = 1$, so the numerator of $P_n(\wp(z))$ must be of degree n^2 , so $\wp(nz) = P_n(\wp(z))$ contains n^2 roots. \square

The grid point visualization provides us with more information. First, it is easy to see that if n is composite, $\wp(nz)$ shares grid-point poles with $\wp(m_i z)$ where m_i are the factors of n . Our $\wp(4z)$ examples illustrates this easily; the halfway points are clearly also poles of $\wp(2z)$. For our polynomial $P_n(\wp(z))$, this means that the denominator of $P_n(\wp(z))$ contains all the factors in the denominator of $P_{m_i}(\wp(z))$.

We also know that $\wp(z)$ is even. On our grid this introduces a rotational symmetry about the center, where pairs of points yield the same value. In our $\wp(z)$ example, $\wp(\frac{3\omega_1 + \omega_2}{4}) = \wp(-\frac{3\omega_1 + \omega_2}{4}) = \wp(\frac{\omega_1 + 3\omega_2}{4})$

For our rational function $P_n(\wp(z))$, because two different grid points z_0 and $-z_0$ have the same value $\wp(z_0)$, this means that the value of $\wp(z_0)$ must be a double pole of the function $P_n(\wp(z))$, thus the denominator of the function factors into squared terms. The only exceptions to this are the halfway points $\frac{\omega_i}{2}$, where $i = 1, 2, 3$ and $\omega_3 = \omega_1 + \omega_2$. It is clear that $\frac{\omega_i}{2}$ and $-\frac{\omega_i}{2}$ are the same grid point, and so the value $\wp(\frac{\omega_i}{2})$ is only a single pole of $P_n(\wp(z))$. This may seem to contradict the fact that if $\frac{\omega_i}{2}$ is on our grid, then it is a double pole of $\wp(nz)$. The reason that this is not a contradiction comes from the fact that $\wp'(\frac{\omega_i}{2}) = 0$, as illustrated in the analytic proof below.

Theorem 3.5. If n is even, the halfway points $\frac{\omega_i}{2}$ are simple poles of $P_n(\wp(z))$, or equivalently:

$$(3.10) \quad \lim_{z \rightarrow \frac{\omega_i}{2}} P_n(\wp(z))(\wp(z) - \wp(\frac{\omega_i}{2})) = c, \text{ for some } c \neq 0$$

Proof. We know that if n is even, $\frac{\omega_i}{2}$ is a double pole of $\wp(nz)$, so we can expand:

$$(3.11) \quad P_n(\wp(z)) = \wp(nz) = \frac{a_{-2}}{(nz - n\frac{\omega_i}{2})^2} + \frac{a_{-1}}{nz - n\frac{\omega_i}{2}} + \sum_{k=0}^{\infty} a_k (nz - n\frac{\omega_i}{2})^k$$

so

$$(3.12)$$

$$\begin{aligned} & \lim_{z \rightarrow \frac{\omega_i}{2}} P_n(\wp(z))(\wp(z) - \wp(\frac{\omega_i}{2})) = \\ & \lim_{z \rightarrow \frac{\omega_i}{2}} \frac{a_{-2}(\wp(z) - \wp(\frac{\omega_i}{2}))}{n^2(z - \frac{\omega_i}{2})} + \frac{a_{-1}(\wp(z) - \wp(\frac{\omega_i}{2}))}{n(z - \frac{\omega_i}{2})} + (\wp(z) - \wp(\frac{\omega_i}{2})) \sum_{k=0}^{\infty} a_k (nz - n\frac{\omega_i}{2})^k \end{aligned}$$

The terms of the sum go to zero, and we may use L'Hopital's Rule on the rational terms:

$$(3.13) \quad = \lim_{z \rightarrow \frac{\omega_i}{2}} \frac{a_{-2}\wp'(z)}{2n^2(z - \frac{\omega_i}{2})} + \frac{a_{-1}\wp'(z)}{n}$$

Because $\wp'(\frac{\omega_i}{2}) = 0$, the term on the right is 0, and we use L'Hopital's Rule again on the term on the left:

$$(3.14) \quad = \lim_{z \rightarrow \frac{\omega_i}{2}} \frac{a_{-2}\wp''(z)}{2n^2} = \frac{a_{-2}\wp''(\frac{\omega_i}{2})}{2n^2} \neq 0$$

□

Theorem 3.6. *Any collection of distinct $P_i(\wp(z))$'s are linearly independent*

Proof. Notice that if a grid point z_0 is not a halfway point $\frac{\omega_i}{2}$, then $\wp'(z_0)$ is not zero and we can use a similar grid argument as before to show that $\wp(z_0)$ is a double pole of $P_n(\wp(z))$. Furthermore because every distinct n gives at least some different grid points, we see that $P_n(\wp(z))$ must have different poles, and so for distinct n_i 's, the $P_{n_i}(\wp(z))$'s are linearly independent. □

Theorem 3.7. *In the expansion of $P_n(\wp(z))$ into a sum of partial fractions,*

$$(3.15) \quad P_n(\wp(z)) = \frac{c_{-2}}{(\wp(z) - \wp(z_0))^2} + \frac{c_{-1}}{\wp(z) - \wp(z_0)} + R_0(\wp(z))$$

where $R_0(\wp(z))$ is some rational function, if z_0 is a pole, such that $z_0 \neq 0, \frac{\omega_i}{2}$, then

$$(3.16) \quad c_{-2} = \frac{\wp'(z_0)^2}{n^2} \text{ and } c_{-1} = \frac{\wp''(z_0)}{n^2}$$

Proof. From the definition of $\wp(nz)$, we see that:

$$(3.17) \quad \wp(nz) = \frac{1}{n^2(z - z_0)^2} - \frac{1}{n^2z_0^2} + \sum_{\omega \in \Gamma, \omega \neq 0, nz_0} \left(\frac{1}{(nz - \omega)^2} - \frac{1}{\omega^2} \right)$$

Now, we set $P_n(\wp(z))$ and $\wp(nz)$ equation to each other, and multiplying the resulting expression by $(z - z_0)^2$:

$$(3.18) \quad \begin{aligned} & \frac{c_{-2}}{\frac{(\wp(z) - \wp(z_0))^2}{(z - z_0)^2}} + \frac{(z - z_0)c_{-1}}{\frac{\wp(z) - \wp(z_0)}{z - z_0}} + (z - z_0)^2 R_0(\wp(z)) \\ &= \frac{1}{n^2} - (z - z_0)^2 \left(\frac{1}{n^2z^2} - \frac{1}{n^2z_0^2} + \sum_{\omega \in \Gamma, \omega \neq 0, nz_0} \left(\frac{1}{(nz - \omega)^2} - \frac{1}{\omega^2} \right) \right) \end{aligned}$$

Taking the limit as $z \rightarrow z_0$ gives:

$$(3.19) \quad \frac{c_{-2}}{\wp'(z_0)^2} = \frac{1}{n^2}$$

And our result for c_{-2} follows.

Now, we attempt to find c_{-1} . Again, we set $P_n(\wp(z)) = \wp(nz)$, but this time we multiply through the expression by $(z - z_0)$ to get:

$$\begin{aligned}
(3.20) \quad & \frac{c_{-2}}{\frac{(\wp(z)-\wp(z_0))^2}{z-z_0}} + \frac{c_{-1}}{\frac{\wp(z)-\wp(z_0)}{z-z_0}} + (z-z_0)R_0(\wp(z)) \\
&= \frac{1}{n^2(z-z_0)} + (z-z_0) \left(\frac{1}{n^2 z^2} - \frac{1}{n^2 z_0^2} + \sum_{\omega \in \Gamma, \omega \neq 0, n z_0} \left(\frac{1}{(nz-\omega)^2} - \frac{1}{\omega^2} \right) \right)
\end{aligned}$$

Moving terms and substituting for c_{-2} gives:

$$\begin{aligned}
(3.21) \quad & \frac{c_{-1}}{\frac{\wp(z)-\wp(z_0)}{z-z_0}} + (z-z_0)R_0(\wp(z)) \\
&= \frac{1}{n^2(z-z_0)} - \frac{\wp'(z_0)^2}{\frac{n^2(\wp(z)-\wp(z_0))^2}{z-z_0}} + (z-z_0) \left(\frac{1}{n^2 z^2} - \frac{1}{n^2 z_0^2} + \sum_{\omega \in \Gamma, \omega \neq 0, n z_0} \left(\frac{1}{(nz-\omega)^2} - \frac{1}{\omega^2} \right) \right)
\end{aligned}$$

Taking the limit as $z \rightarrow z_0$ yields:

$$(3.22) \quad \frac{c_{-1}}{\wp'(z_0)} = \frac{1}{n^2} \lim_{z \rightarrow z_0} \left(\frac{1}{z-z_0} - \frac{\wp'(z_0)^2}{\frac{(\wp(z)-\wp(z_0))^2}{z-z_0}} \right)$$

Because $z_0 \neq 0$, we can substitute the Taylor series expansion for $\wp(z)$:

$$(3.23) \quad \wp(z) = \wp(z_0) + \wp'(z_0)(z-z_0) + \frac{\wp''(z_0)}{2!}(z-z_0)^2 + \dots$$

Substitute into our expression, giving

$$\begin{aligned}
\frac{n^2 c_{-1}}{\wp'(z_0)} &= \lim_{z \rightarrow z_0} \left(\frac{1}{z-z_0} - \frac{\wp'(z_0)^2}{\frac{(\wp(z_0) + \wp'(z_0)(z-z_0) + \frac{\wp''(z_0)}{2!}(z-z_0)^2 + \dots - \wp(z_0))^2}{z-z_0}} \right) \\
&= \lim_{z \rightarrow z_0} \left(\frac{1}{z-z_0} - \frac{\wp'(z_0)^2}{\frac{((z-z_0)(\wp'(z_0) + \frac{\wp''(z_0)}{2!}(z-z_0) + \dots))^2}{z-z_0}} \right) \\
&= \lim_{z \rightarrow z_0} \left(\frac{(\wp'(z_0) + \frac{\wp''(z_0)}{2!}(z-z_0) + \dots)^2 - \wp'(z_0)^2}{(z-z_0)(\wp'(z_0) + \frac{\wp''(z_0)}{2!}(z-z_0) + \dots)^2} \right) \\
&= \lim_{z \rightarrow z_0} \left(\frac{\wp'(z_0)^2 + 2(z-z_0)\wp'(z_0)\frac{\wp''(z_0)}{2!} + \dots - \wp'(z_0)^2}{(z-z_0)(\wp'(z_0) + \frac{\wp''(z_0)}{2!}(z-z_0) + \dots)^2} \right) \\
&= \lim_{z \rightarrow z_0} \left(\frac{\wp'(z_0)\wp''(z_0) + \dots}{(\wp'(z_0) + \frac{\wp''(z_0)}{2!}(z-z_0) + \dots)^2} \right) \\
&= \frac{\wp'(z_0)\wp''(z_0)}{\wp'(z_0)^2}
\end{aligned}$$

□

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