PROPERTIES OF THE ITERATES OF THE WEIERSTRASS-\(\wp\) FUNCTION

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This paper is dedicated to our authors.

Abstract. We shall here discuss several properties of the Weierstrass-\(\wp\) function, as defined on the fundamental parallelogram \(\mathbb{C}/\Gamma\), where \(\mathbb{C}\) is the complex plane and \(\Gamma\) is the lattice generated by \(\omega_1\) and \(\omega_2\). Using the addition formula for \(\wp(z_1 + z_2)\), we shall develop a recurrence relation for \(\wp(nz)\) in terms of \(\wp(z)\). We shall examine the degree of this expression, some coefficients, and patterns concerning the poles of this function. We shall also consider the geometric interpretation of taking an arbitrary \(z_0\) and adding it to itself, both in the fundamental parallelogram \(\mathbb{C}/\Gamma\) and the elliptic curve generated by \(\wp(z)\) and \(\wp'(z)\).

1. Preliminaries

1.1. Chebyshev Polynomials.

Definition 1.1. The first-kind Chebyshev Polynomial \(T_n(x)\) is a polynomial in \(x\) of degree \(n\) defined by:

\[
T_n(x) = \frac{1}{2} [\cos(n\theta) \cos(n\theta) - \cos((n-1)\theta)]
\]

where \(x = \cos \theta\)

Theorem 1.2. The fundamental recurrence relation that generates all of the polynomials \(T_n(x)\) is as follows:

\[
T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \text{ for } n = 2, 3, \ldots
\]

with initial conditions

\[
T_0(x) = 1 \text{ and } T_1(x) = x
\]

The nice properties of the Chebyshev Polynomials lead us to develop a recurrence relation for \(\wp(nz)\) and look at the properties of the resulting explicit rational function.

1.2. Weierstrass-\(\wp\) Function.

Definition 1.3. Let points \(\omega_1, \omega_2 \in \mathbb{C}\) have non-colinear position vectors. Now define the lattice \(\Gamma\) as all points of the form

\[
\omega \in \Gamma \iff \omega = a\omega_1 + b\omega_2 \quad \forall a, b \in \mathbb{Z}
\]
The Weierstrass-$\wp$ function is defined as:

\begin{equation}
\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma, \omega \neq 0} \left( \frac{1}{z - \omega} - \frac{1}{\omega^2} \right) \quad \forall z \in \mathbb{C}
\end{equation}

**Remark 1.4.** Notice that $\wp(z)$ is even, and more importantly, that $\wp(z)$ is double periodic, with periods of $\omega_1$ and $\omega_2$, i.e.

\begin{equation}
\wp(z + \omega_1) = \wp(z + \omega_2) = \wp(z) \quad \forall z \in \mathbb{C}
\end{equation}

Because of this, we may examine $\wp(z)$ on a fundamental parallelogram $\mathbb{C}/\Gamma$ defined by $\omega_1$ and $\omega_2$. Notice that $\wp(z)$ contains its only poles at lattice points, and these are order two poles.

**Theorem 1.5.** Using power series expansions and pole/zero comparisons, we see that

\begin{equation}
\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3
\end{equation}

where

\begin{equation}
g_2 = 60 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^4} \quad \text{and} \quad g_3 = 140 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^6}
\end{equation}

and that

\begin{equation}
\wp'(z) = -2z^3 + 6 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^3} z + 20 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^6} z^3
\end{equation}

**Remark 1.6.** From here we see that $\wp'(z)$ is also double-periodic on our lattice, and that it is odd. Because of this, we see that the half-periods must be zeroes of $\wp'(z)$. In our fundamental parallelogram, we have:

\begin{equation}
\wp'\left(\frac{\omega_2}{2}\right) = \wp'\left(\frac{\omega_2}{2}\right) = \wp'\left(\frac{\omega_1 + \omega_2}{2}\right) = 0.
\end{equation}

We also see that the points $(\wp(z), \wp'(z))$ lie on the curve $y^2 = 4x^3 - g_2x - g_3$. This is an elliptic curve with roots at $x = \wp\left(\frac{\omega_1}{2}\right), \wp\left(\frac{\omega_2}{2}\right), \wp\left(\frac{\omega_1 + \omega_2}{2}\right)$.

Using the relationship between $\wp(z)$ and this curve, we can now begin to derive a recurrence relation.

### 2. Recurrence Relation for $\wp(nz)$

**Theorem 2.1.** For $z_1, z_2 \in \mathbb{C},$

\begin{equation}
\wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4} \left( \frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)} \right)^2
\end{equation}

If $z_1 \equiv z_2 \mod \Gamma$, the limit as $z_2 \to z_1$ gives

\begin{equation}
\wp(2z_1) = -2\wp(z_1) + \frac{1}{4} \left( \frac{\wp''(z_1)}{\wp'(z_1)} \right)^2
\end{equation}

Now we can give a recurrence relation for $\wp(nz)$:

**Theorem 2.2.**
3. Basic properties of \( \wp(nz) \)

**Theorem 3.1.**

\[
\wp(nz) = P_n(\wp(z)) = \frac{1}{n^2} + R_n(\wp(z))
\]

where \( R_n \) is some rational function of \( \wp(z) \) with degree less than 0, \( \forall n \in \mathbb{Z} \)

**Proof.** We write the power series expansion for \( \wp(z) \) at the origin:

\[
\wp(z) = \frac{1}{z^2} + 3 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^4} z^2 + 5 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^6} z^4 + ...
\]

so

\[
\wp(nz) = \frac{1}{n^2 z^2} + 3 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^4} n^2 z^2 + 5 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^6} n^4 z^4 + ...
\]

Define:

\[
H_n(z) = 3 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^4} n^2 z^2 + 5 \sum_{\omega \in \Gamma, \omega \neq 0} \frac{1}{\omega^6} n^4 z^4 + ...
\]

so

\[
\wp(nz) = \frac{1}{n^2 z^2} + H_n(z)
\]

Notice that as \( z \to 0 \), \( H_n(z) \to 0 \) as well. We also know that \( \wp(nz) = P_n(\wp(z)) \).

Substitute in our expansion:

\[
\wp(nz) = P_n \left( \frac{1}{z^2} + H_1(z) \right)
\]

Since \( P_n(x) \) is some rational function of \( x \), define \( \tilde{P}_n(x) \) as the quotient with no term of degree less than 0, and \( R_n(x) \) as the remainder of degree less than 0, so:

\[
\wp(nz) = P_n \left( \frac{1}{z^2} + H_1(z) \right) = \tilde{P}_n \left( \frac{1}{z^2} + H_1(z) \right) + R_n \left( \frac{1}{z^2} + H_1(z) \right)
\]

\[
= \frac{1}{n^2 z^2} + H_n(z)
\]

First note that as \( z \to 0 \), \( (\frac{1}{z^2} + H_1(z)) \to \infty \). But since \( \deg R_n < 0 \), \( R_n(\frac{1}{z^2} + H_1(z)) \to 0 \), so \( R_n \) will not contribute to the \( \frac{1}{z^2} \) term. As for the \( \tilde{P}_n \) term,

\[
\tilde{P}_n \left( \frac{1}{z^2} + H_1(z) \right) = a_0 + a_1 \left( \frac{1}{z^2} + H_1(z) \right) + a_2 \left( \frac{1}{z^2} + H_1(z) \right)^2 + ...
\]

Because \( R_n \) does not contribute in the \( \frac{1}{z^2} \) term, we see that \( a_1 = \frac{1}{n^2} \), and because the expression for \( \wp(nz) \) contains no constant terms and no terms of degree less than 2, we see that \( a_0 = a_2 = a_3 = 0 \), so,

\[
P_n(\wp(z)) = \tilde{P}_n(\wp(z)) + R_n(\wp(z))
\]

\[
= \frac{1}{n^2} \wp(z) + R_n(\wp(z))
\]

\( \square \)

**Remark 3.2.** From this we see that the rational function \( P_n(\wp(z)) \) is always of degree 1
Lemma 3.3. \( z_0 \) is a pole of \( \varphi(nz) \) if and only if \( nz_0 \in \Gamma \), or \( nz_0 = a\omega_1 + b\omega_2 \) so \( z_0 = \frac{a\omega_1 + b\omega_2}{n} \) for some \( a, b \in \mathbb{Z} \)

Theorem 3.4. \( \varphi(nz) = P_n(\varphi(z)) \) contains \( n^2 \) roots

Proof. In the above lemma, clearly both \( a \) and \( b \) can take any integer value from 0 to \( n-1 \), so there are \( n^2 \) poles of \( \varphi(nz) \).

But as \( \varphi(nz) = P_n(\varphi(z)) \), then if a point \( z_0 \) is a pole of \( \varphi(nz) \), then \( \varphi(z_0) \) must be a pole of \( P_n(\varphi(z)) \), which means that \( \varphi(z_0) \) is a root of the denominator of the rational function \( P_n(\varphi(z)) \). The one exception is the point \( z_0 = 0 \), because \( \varphi(0) = \infty \), which is not the root of any polynomial. Since \( \varphi(z) \) is well defined everywhere but on the lattice points, this means that the denominator of \( P_n(\varphi(z)) \) must have \( n^2 - 1 \) roots, or must be of degree \( n^2 - 1 \). We proved earlier that \( \deg P_n = 1 \), so the numerator of \( P_n(\varphi(z)) \) must be of degree \( n^2 \), so \( \varphi(nz) = P_n(\varphi(z)) \) contains \( n^2 \) roots.

The grid point visualization provides us with more information. First, it is easy to see that if \( n \) is composite, \( \varphi(nz) \) shares grid-point poles with \( \varphi(m_i z) \) where \( m_i \) are the factors of \( n \). Our \( \varphi(4z) \) examples illustrate this easily; the halfway points are clearly also poles of \( \varphi(2z) \). For our polynomial \( P_n(\varphi(z)) \), this means that the denominator of \( P_n(\varphi(z)) \) contains all the factors in the denominator of \( P_{m_i}(\varphi(z)) \).

We also know that \( \varphi(z) \) is even. On our grid this introduces a rotational symmetry about the center, where pairs of points yield the same value. In our \( \varphi(z) \) example, \( \varphi\left(\frac{3\omega_1 + 3\omega_2}{2}\right) = \varphi\left(-\frac{3\omega_1 + 3\omega_2}{2}\right) = \varphi\left(\frac{\omega_1 + 3\omega_2}{2}\right) \)

For our rational function \( P_n(\varphi(z)) \), because two different grid points \( z_0 \) and \( -z_0 \) have the same value \( \varphi(z_0) \), this means that the value of \( \varphi(z_0) \) must be a double pole of the function \( P_n(\varphi(z)) \), thus the denominator of the function factors into squared terms. The only exceptions to this are the halfway points \( \frac{\omega_i}{2} \), where \( i = 1, 2, 3 \) and \( \omega_3 = \omega_1 + \omega_2 \). It is clear that \( \frac{\omega_i}{2} \) and \( \frac{-\omega_i}{2} \) are the same grid point, and so the value \( \varphi\left(\frac{\omega_i}{2}\right) \) is only a single pole of \( P_n(\varphi(z)) \). This may seem to contradict the fact that if \( \frac{\omega_i}{2} \) is on our grid, then it is a double pole of \( \varphi(nz) \). The reason that this is not a contradiction comes from the fact that \( \varphi'\left(\frac{\omega_i}{2}\right) = 0 \), as illustrated in the analytic proof below.

Theorem 3.5. If \( n \) is even, the halfway points \( \frac{\omega_i}{2} \) are simple poles of \( P_n(\varphi(z)) \), or equivalently:

\[
\lim_{z \to \frac{\omega_i}{2}} P_n(\varphi(z))(\varphi(z) - \varphi\left(\frac{\omega_i}{2}\right)) = c, \quad \text{for some } c \neq 0
\]

Proof. We know that if \( n \) is even, \( \frac{\omega_i}{2} \) is a double pole of \( \varphi(nz) \), so we can expand:

\[
P_n(\varphi(z)) = \varphi(nz) = \frac{a_{-2}}{(nz - n\frac{\omega_i}{2})^2} + \frac{a_{-1}}{nz - n\frac{\omega_i}{2}} + \sum_{k=0}^{\infty} a_k(nz - n\frac{\omega_i}{2})^k
\]

so

\[
\lim_{z \to \frac{\omega_i}{2}} P_n(\varphi(z))(\varphi(z) - \varphi\left(\frac{\omega_i}{2}\right)) = \lim_{z \to \frac{\omega_i}{2}} \frac{a_{-2}(\varphi(z) - \varphi\left(\frac{\omega_i}{2}\right))}{n^2(z - \frac{n\omega_i}{2})} + \frac{a_{-1}(\varphi(z) - \varphi\left(\frac{\omega_i}{2}\right))}{n(z - \frac{n\omega_i}{2})} + (\varphi(z) - \varphi\left(\frac{\omega_i}{2}\right)) \sum_{k=0}^{\infty} a_k(nz - n\frac{\omega_i}{2})^k
\]
The terms of the sum go to zero, and we may use L’Hopital’s Rule on the rational terms:

\[
(3.13) \quad \lim_{z \to \frac{a}{2n^2}} \frac{a-2\psi'(z)}{2n^2(z-a)} + \frac{a-1\psi'(z)}{n}
\]

Because \(\psi'\left(\frac{a}{2n^2}\right) = 0\), the term on the right is 0, and we use L’Hopital’s Rule again on the term on the left:

\[
(3.14) \quad \lim_{z \to \frac{a}{2n^2}} \frac{a-2\psi''(z)}{2n^2} = \frac{a-2\psi''\left(\frac{a}{2n^2}\right)}{2n^2} \neq 0
\]

\(\square\)

**Theorem 3.6.** Any collection of distinct \(P_i(\psi(z))\)'s are linearly independent

**Proof.** Notice that if a grid point \(z_0\) is not a halfway point \(\frac{a}{2n}\), then \(\psi'(z_0)\) is not zero and we can use a similar grid argument as before to show that \(\psi(z_0)\) is a double pole of \(P_n(\psi(z))\). Furthermore because every distinct \(n\) gives at least some different grid points, we see that \(P_n(\psi(z))\) must have different poles, and so for distinct \(n\)'s, the \(P_n(\psi(z))\)'s are linearly independent. \(\square\)

**Theorem 3.7.** In the expansion of \(P_n(\psi(z))\) into a sum of partial fractions,

\[
(3.15) \quad P_n(\psi(z)) = \frac{c_{-2}}{(\psi(z) - \psi(z_0))^2} + \frac{c_{-1}}{\psi(z) - \psi(z_0)} + R_0(\psi(z))
\]

where \(R_0(\psi(z))\) is some rational function, if \(z_0\) is a pole, such that \(z_0 \neq 0, \frac{a}{2n}\), then

\[
(3.16) \quad c_{-2} = \frac{\psi'(z_0)^2}{n^2} \text{ and } c_{-1} = \frac{\psi''(z_0)}{n^2}
\]

**Proof.** From the definition of \(\psi(nz)\), we see that:

\[
(3.17) \quad \psi(nz) = \frac{1}{n^2(z-z_0)^2} - \frac{1}{n^2z_0^2} + \sum_{\omega \in \Gamma, \omega \neq 0, nz_e} \left( \frac{1}{(nz-\omega)^2} - \frac{1}{\omega^2} \right)
\]

Now, we set \(P_n(\psi(z))\) and \(\psi(nz)\) equation to each other, and multiplying the resulting expression by \((z-z_0)^2\):

\[
(3.18) \quad = \frac{c_{-2}}{\psi(z) - \psi(z_0)} + \frac{(z-z_0)c_{-1}}{(z-z_0)} + (z-z_0)^2R_0(\psi(z))
\]

\[
\frac{1}{n^2} - (z-z_0)^2 \left( \frac{1}{n^2z_0^2} - \frac{1}{n^2z_0^2} + \sum_{\omega \in \Gamma, \omega \neq 0, nz_e} \left( \frac{1}{(nz-\omega)^2} - \frac{1}{\omega^2} \right) \right)
\]

Taking the limit as \(z \to z_0\) gives:

\[
(3.19) \quad \frac{c_{-2}}{\psi'(z_0)^2} = \frac{1}{n^2}
\]

And our result for \(c_{-2}\) follows.

Now, we attempt to find \(c_{-1}\). Again, we set \(P_n(\psi(z)) = \psi(nz)\), but this time we multiply through the expression by \((z-z_0)^2\) to get:
\[(3.20)\]
\[
\frac{c-2}{(\varphi(z) - \varphi(z_0))^2} + \frac{c-1}{(\varphi(z) - \varphi(z_0))} + (z - z_0)R_0(\varphi(z))
\]
\[
= \frac{1}{n^2(z - z_0)} + (z - z_0) \left( \frac{1}{n^2 z^2} - \frac{1}{n^2 z_0^2} + \sum_{\omega \in \Gamma, \omega \neq 0, n z_0} \left( \frac{1}{(nz - \omega)^2} - \frac{1}{\omega^2} \right) \right)
\]

Moving terms and substituting for \(c-2\) gives:

\[(3.21)\]
\[
\frac{c-1}{(\varphi(z) - \varphi(z_0))} + (z - z_0)R_0(\varphi(z))
\]
\[
= \frac{1}{n^2(z - z_0)} - \frac{\varphi'(z_0)^2}{n^2(\varphi(z) - \varphi(z_0))^2} + (z - z_0) \left( \frac{1}{n^2 z^2} - \frac{1}{n^2 z_0^2} + \sum_{\omega \in \Gamma, \omega \neq 0, n z_0} \left( \frac{1}{(nz - \omega)^2} - \frac{1}{\omega^2} \right) \right)
\]

Taking the limit as \(z \to z_0\) yields:

\[(3.22)\]
\[
\frac{c-1}{\varphi'(z_0)} = \frac{1}{n^2 z - z_0} \lim_{z \to z_0} \left( \frac{1}{z - z_0} - \frac{\varphi'(z_0)^2}{(\varphi(z) - \varphi(z_0))^2} \right)
\]

Because \(z_0 \neq 0\), we can substitute the Taylor series expansion for \(\varphi(z)\):

\[(3.23)\]
\[
\varphi(z) = \varphi(z_0) + \varphi'(z_0)(z - z_0) + \frac{\varphi''(z_0)}{2!}(z - z_0)^2 + ...
\]

Substitute into our expression, giving

\[
\frac{n^2c-1}{\varphi'(z_0)} = \lim_{z \to z_0} \left( \frac{1}{z - z_0} - \frac{\varphi'(z_0)^2}{(\varphi(z) + \varphi'(z_0)(z - z_0) + \frac{\varphi''(z_0)}{2!}(z - z_0)^2 + ... - \varphi(z_0))^2} \right)
\]

\[
= \lim_{z \to z_0} \left( \frac{1}{z - z_0} - \frac{\varphi'(z_0)^2}{(z - z_0)(\varphi'(z_0) + \frac{\varphi''(z_0)}{2!}(z - z_0) + ... - \varphi(z_0))^2} \right)
\]

\[
= \lim_{z \to z_0} \left( \frac{\varphi'(z_0)^2 + 2(z - z_0)\varphi'(z_0)\frac{\varphi''(z_0)}{2!}(z - z_0) + ... - \varphi'(z_0)^2}{(z - z_0)(\varphi'(z_0) + \frac{\varphi''(z_0)}{2!}(z - z_0) + ...)^2} \right)
\]

\[
= \lim_{z \to z_0} \left( \frac{\varphi'(z_0)\varphi''(z_0) + ...}{(\varphi(z_0) + \frac{\varphi''(z_0)}{2!}(z - z_0) + ...)^2} \right)
\]

\[
= \frac{\varphi'(z_0)\varphi''(z_0)}{\varphi'(z_0)^2}
\]
REFERENCES


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