

# Blocking Strategies for a Fire Control Problem

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**Abstract.** In this paper we analyze different strategies, in a problem of optimal confinement of a forest fire. The area burned by the fire at time  $t > 0$  is modelled as the reachable set for a differential inclusion  $\dot{x} \in F(x)$ , starting from an initial set  $R_0$ . To encircle the fire, a wall can be constructed progressively in time, at a given speed. We examine the minimum construction speed which is needed to completely encircle the fire, by means of one single wall. Different strategies are then compared, by a theoretical analysis and by numerical experiments, to determine which one minimizes the total burned area. We consider first the isotropic case, where the fire propagates uniformly in all directions, then a more general case, where the wind blows the fire in one preferred direction.

## 1 - Introduction

The mathematical problems considered in this paper are motivated by the control of forest fires. More generally, our models describe the spatial spreading of a contaminating agent, which a controller wishes to block by constructing a barrier, in real time.

We review here the main features of the model, introduced in [3]. The set burned by the fire (or reached by the contamination) at time  $t \geq 0$  is denoted by  $R(t) \subset \mathbb{R}^2$ . In absence of any control, the growth of this set is determined by a differential inclusion. Let  $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$  be a Lipschitz continuous multifunction with compact, convex values, and consider an initial set  $R_0 \subset \mathbb{R}^2$ . We then define  $R(t)$  as the reachable set for the differential inclusion

$$\dot{x} \in F(x) \quad x(0) \in R_0, \quad (1.1)$$

where the upper dot denotes a derivative w.r.t. time. Equivalently,

$$R(t) = \left\{ x(t); \ x(\cdot) \text{ absolutely continuous, } x(0) \in R_0, \ \dot{x}(\tau) \in F(x(\tau)) \text{ for a.e. } \tau \in [0, t] \right\}. \quad (1.2)$$

A comprehensive introduction to the theory of differential inclusions can be found in [1] or in [6]. We assume that

$$0 \in F(x) \quad \text{for all } x \in \mathbb{R}^2, \quad (1.3)$$

hence

$$R(t_1) \subseteq R(t_2) \quad \text{whenever } t_1 \leq t_2. \quad (1.4)$$

In our model, the spreading of the contamination can be controlled by erecting walls, or barriers. In the case of a forest fire, one may think of a thin strip of land which is either soaked with water poured from above (by an airplane or a helicopter), or cleared from all vegetation using a bulldozer. In any case, this will prevent the fire from crossing that particular strip of land.

**Definition 1.** Let a constant  $\sigma > 0$  be given. We say that a set valued map  $t \mapsto \gamma(t) \subset \mathbb{R}^2$  is an **admissible strategy**, and write  $\gamma \in \mathcal{A}$ , if the following two conditions hold:

- (C1) For every  $t_1 \leq t_2$  one has  $\gamma(t_1) \subseteq \gamma(t_2)$ .
- (C2) Each  $\gamma(t)$  is a rectifiable curve with length  $\leq \sigma t$ .

Here  $\sigma$  is the speed at which walls can be constructed. Notice that we never require that the sets  $\gamma(t)$  be connected. For example, for each  $t > 0$ , the curve  $\gamma(t) = \gamma_1(t) \cup \gamma_2(t)$  could be the union of two smooth arcs, having length  $\sigma t/4$  and  $3\sigma t/4$ , respectively.

By constructing walls, the controller can restrict the growth of the reachable set. Indeed, adopting the strategy  $\gamma$ , the corresponding reachable set is defined as

$$R^\gamma(t) \doteq \left\{ x(t); \begin{array}{l} x(\cdot) \text{ absolutely continuous, } x(0) \in R_0, \dot{x}(\tau) \in F(x(\tau)) \text{ for a.e. } \tau \in [0, t], \\ x(\tau) \notin \gamma(\tau) \text{ for all } \tau \in [0, t] \end{array} \right\}. \quad (1.5)$$

In other words,  $R^\gamma(t)$  is the set reached by trajectories of the differential inclusion (1.1) which, at any given time  $\tau$ , do not cross the previously constructed walls.

An interesting question is whether there exists an admissible strategy  $\gamma(\cdot)$  which completely encircles the fire, eventually stopping its propagation. Recalling (1.5), this can be formulated as:

**Blocking problem.** Find conditions on the multifunction  $F$  and on the speed  $\sigma$  which provide the existence of an admissible strategy  $t \mapsto \gamma(t) \subset \mathbb{R}^2$  blocking the fire. Namely,

$$R^\gamma(t) \subseteq B_r \quad \text{for all } t > 0, \quad (1.6)$$

for some fixed ball  $B_r$  centered at the origin with radius  $r$ .

In the case where at least one confining strategy exists, one can search for an optimal one, minimizing the total area of the region burned by the fire. Calling

$$R_\infty^\gamma \doteq \bigcup_{t \geq 0} R^\gamma(t), \quad (1.7)$$

this leads to the following:

**Optimization problem.** Find an admissible strategy  $\gamma \in \mathcal{A}$  for which the two-dimensional Lebesgue measure of the burned set is the smallest possible. In other words,

$$\text{minimize: } \text{meas}(R_\infty^\gamma). \quad (1.8)$$

A general result on the existence of optimal strategies will be given in [4]. It is important to notice that in all these problems the wall must be constructed “in real time”. The construction of the wall and the propagation of the fire occur simultaneously. In particular, finding a closed curve  $\Gamma$  of minimum length which completely encircles the initial set  $R_0$  is useless for our purposes. Indeed, by the time needed to construct a wall along  $\Gamma$ , the fire may have already spread far beyond the region bounded by this curve.

To simplify our analysis, we assume that the set of propagation velocities is independent of  $x$ , so that  $F(x) \equiv F$  for a given compact convex set  $F \subset \mathbb{R}^2$ . We shall consider two main cases.

- (i) The isotropic case, where the fire propagates with unit speed in all directions. After a rescaling of coordinates, this amounts to

$$F = B_1, \tag{1.9}$$

where  $B_1$  is the closed unit ball centered at the origin.

- (ii) The non-isotropic case, assuming that the set of propagation velocities  $F$  has the form

$$F = \left\{ (r \cos \theta, r \sin \theta); 0 \leq r \leq \rho(\theta) \right\}, \tag{1.10}$$

where the function  $\rho : [-\pi, \pi] \mapsto \mathbb{R}_+$  satisfies

$$\rho(-\theta) = \rho(\theta), \quad 0 \leq \rho(\theta') \leq \rho(\theta) \quad \text{for all } 0 \leq \theta \leq \theta' \leq \pi. \tag{1.11}$$

The more general case (1.10) models the presence of wind, which blows the fire with greater speed in a given direction.

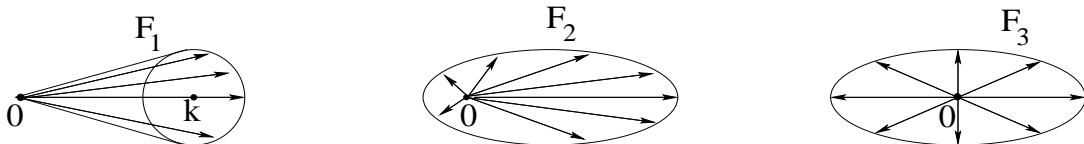


Figure 1. Three examples of non-isotropic velocity sets. The sets  $F_1$  and  $F_2$  satisfy the assumptions (1.11), while  $F_3$  does not.

The remainder of this paper is organized as follows. Section 2 contains a brief review of some basic definitions, an elementary comparison argument and a useful rescaling property of our equations. In Section 3 we determine conditions on the speed  $\sigma$  which allow the complete encirclement of the fire by means of a spiral-like wall. In particular, this provides a new answer to the fire blocking problem, imposing that wall construction occurs only along one end-point. This models a situation where one single team of firemen is available. The result proved in [3], on the other hand, required two teams simultaneously working at two different places. In Section 4 we compare the total area burned by the fire when different strategies are implemented. Walls are here constructed along one, two, or more spirals. Letting the construction speed  $\sigma \rightarrow \infty$ , we can give an estimate of the area of the region  $R_\infty^\gamma$ , by means of an expansion in powers of  $\varepsilon = \sigma^{-1}$ , and compare the performance of different strategies. In Section 5 we study the case of non-isotropic

fire propagation, described by (1.10). We derive conditions on the function  $\psi$  and on the wall construction speed  $\sigma$  which allow a complete encircling of the fire. Section 6 contains the results of numerical experiments, computing the areas of the sets  $R_\infty^\gamma$  for various values of  $\sigma$ . Here we first use a double spiral strategy, then we compare it with the optimal strategy, consisting of an arc of circumference plus two spirals. We also perform a similar comparison, for a case of non-isotropic fire propagation speed.

## 2 - Preliminaries

We briefly review here some results and definitions from [3]. Throughout the following,  $\text{int } S$  and  $\bar{S}$  denote the interior and the closure of a set  $S \subset \mathbb{R}^2$ , respectively.

The following simple comparison result is often useful. Consider two multifunctions  $F, \tilde{F}$  and two initial sets  $R_0, \tilde{R}_0$ . If  $R_0 \subseteq \tilde{R}_0$  and  $F(x) \subseteq \tilde{F}(x)$  for every  $x$ , then for every blocking strategy  $t \mapsto \gamma(t)$  the corresponding sets reached by the fire satisfy

$$R^\gamma(t) \subseteq \tilde{R}^\gamma(t) \quad \text{for all } t \geq 0. \quad (2.1)$$

Next, we observe that, if the velocity sets  $F(x)$  are independent of  $x$ , then our model is invariant under space and time rescalings. Namely, fix a constant  $\lambda > 0$  and consider an admissible strategy  $t \mapsto \gamma(t)$ , satisfying the conditions (C1)-(C2). Then the strategy  $t \mapsto \tilde{\gamma}(t) \doteq \lambda \gamma(t/\lambda)$  is also admissible. If  $R^\gamma(t)$  are the sets reached by the fire starting from  $R_0$ , in connection with the strategy  $\gamma$ , then

$$\tilde{R}(t) = \lambda R^\gamma(t/\lambda) = \{\lambda x; \ x \in R^\gamma(t/\lambda)\} \quad (2.2)$$

are the sets reached by the fire at time  $t$ , starting from  $\tilde{R}_0 = \lambda R_0$ , when the strategy  $\tilde{\gamma}$  is implemented. A useful consequence of this observation is

**Lemma 1.** *Let the velocity sets  $F(x)$  be independent of  $x$ . Assume that there exists an open set  $R_0^*$  such that the fire starting from  $R_0^*$  can be completely confined by some admissible strategy  $\gamma^*(\cdot)$ . Then, for every bounded set  $R_0$ , there exists a strategy  $\gamma(\cdot)$  that blocks a fire starting from  $R_0$ .*

Indeed, by a translation of coordinates we can assume that  $R_0^*$  is a neighborhood of the origin. For any given  $R_0$ , we can find  $\lambda > 0$  large enough so that

$$R_0 \subseteq \lambda R_0^*.$$

Then the strategy  $\gamma(t) = \lambda \gamma^*(t/\lambda)$  is admissible and blocks a fire starting from  $\lambda R_0^*$ , hence also a fire starting from the smaller set  $R_0$ .  $\square$

Optimal strategies usually consist of a concatenation of finitely many walls. These can be of different types. More precisely, consider a strategy  $\gamma$  which blocks the fire within time  $T$ . This means that, at time  $t = T$ , the fire is completely encircled by the wall, and no construction takes place afterwards, i.e.  $\gamma(t) = \gamma(T)$  for all  $t \geq T$ . For  $0 \leq t < T$  we call

$$\partial\gamma(t) \doteq \bigcap_{\tau > t} \overline{\gamma(\tau) \setminus \gamma(t)} = \{x_1(t), \dots, x_\nu(t)\} \quad (2.3)$$

the points along the set of walls  $\gamma(t)$  where new construction is taking place. Arcs in  $\gamma(T)$  are classified in two groups:  $\mathcal{F}$  and  $\mathcal{B}$  (Free arcs and Boundary arcs).

- Free arcs are those which, at the time when they are constructed, lie away from the contaminated region. More precisely, we say that a regular arc  $\Gamma \subseteq \gamma(T)$  is a **free arc** if

$$\Gamma \cap \partial\gamma(t) \cap \overline{R^\gamma(t)} = \emptyset \quad \text{for all } t \in [0, T]. \quad (2.4)$$

- Boundary arcs are those which, at the time when they are constructed, lie on the boundary the contaminated region. More precisely, we say that a regular arc  $\Gamma \subseteq \gamma(T)$  is a **boundary arc** if

$$(\Gamma \cap \partial\gamma(t)) \subset \overline{R^\gamma(t)} \quad \text{for all } t \in [0, T]. \quad (2.5)$$

We write  $\Gamma_j \in \mathcal{F}$  or  $\Gamma_j \in \mathcal{B}$  in the case where (2.4) or (2.5) holds, respectively.

### 3 - The single spiral strategy

Thanks to the rescaling argument described at (2.2) and in Lemma 1, it suffices to study the blocking problem in the special case where the initial set coincides with the unit disc centered at the origin, i.e.  $R_0 = B_1$ . Moreover, we assume that  $F(x) \equiv B_1$ , so that the fire propagates in all directions with unit speed. In this section, we investigate for which values of the wall construction speed  $\sigma$  one can completely encircle the fire by a single spiral-like wall. More precisely, we start from the point  $P = (1, 0)$  and construct a wall with speed  $\sigma$ , in such a way that the burned region always remains within the inner side of the constructed wall. If  $\sigma > 1$  is sufficiently large, we will show that this spiral-like wall can be closed on itself within finite time, completely encircling the fire.

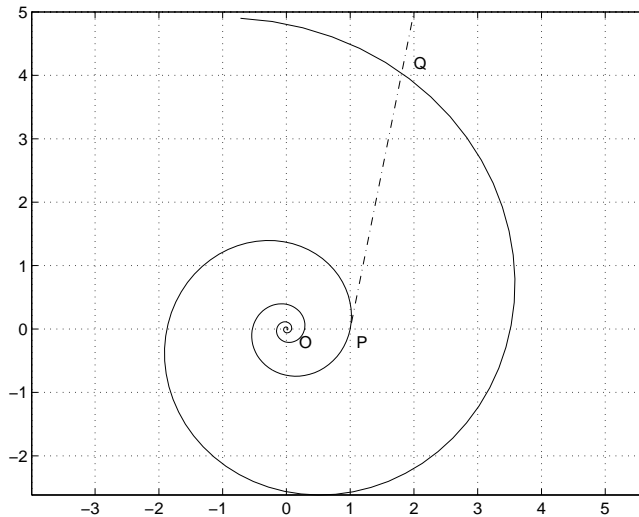


Figure 2. The spiral defining the function  $\psi$ , in the case  $\lambda = 0.2$ .

To determine how large the speed  $\sigma$  should be, a preliminary construction is needed. For each  $\lambda > 0$ , consider the spiral which in polar coordinates has equation  $r = e^{\lambda\theta}$ , namely

$$S^\lambda \doteq \{(e^{\lambda\theta} \cos \theta, e^{\lambda\theta} \sin \theta), \quad \theta \in \mathbb{R}\} \subset \mathbb{R}^2.$$

From the point  $P = (1, 0)$  we send a tangent line to  $S^\lambda$ . This will intersect  $S^\lambda$  at a second point  $Q$ . We now define

$$\psi(\lambda) \doteq \frac{[\text{length of the arc of spiral between } 0 \text{ and } Q]}{[\text{length of the arc of spiral between } 0 \text{ and } P] + [\text{length of the segment } PQ]}.$$

We observe that

$$\psi(\lambda) > 1 \quad \text{for all } \lambda > 0, \quad (3.1)$$

because moving from  $P$  to  $Q$  along the spiral takes certainly longer than moving along a straight segment. Moreover,

$$Q = (e^{\lambda\Theta} \cos \Theta, e^{\lambda\Theta} \sin \Theta)$$

for some angle  $\Theta = \Theta(\lambda) \in [2\pi, 5\pi/2]$ , implicitly defined by the relation

$$e^{\lambda\Theta} \cos \Theta - 1 = \lambda e^{\lambda\Theta} \sin \Theta. \quad (3.2)$$

Using (3.2), a direct computation yields

$$\begin{aligned} \psi(\lambda) &= \frac{(1/\lambda)e^{\lambda\Theta}\sqrt{1+\lambda^2}}{(1/\lambda)\sqrt{1+\lambda^2} + \sqrt{(e^{\lambda\Theta} \cos \Theta - 1)^2 + e^{2\lambda\Theta} \sin^2 \Theta}} \\ &= \frac{e^{\lambda\Theta}}{1 + \lambda e^{\lambda\Theta} \sin \Theta} = \frac{1}{\cos \Theta}. \end{aligned} \quad (3.3)$$

As  $\lambda \rightarrow 0$ , by (3.3) we have

$$\limsup_{\lambda \rightarrow 0^+} \psi(\lambda) \leq \limsup_{\lambda \rightarrow 0^+} e^{\lambda\Theta} \leq \lim_{\lambda \rightarrow 0} e^{5\pi\lambda/2} = 1. \quad (3.4)$$

Next, from (3.2) we deduce

$$\tan \Theta(\lambda) < \frac{1}{\lambda},$$

hence  $\lim_{\lambda \rightarrow \infty} \Theta(\lambda) = 2\pi$ . From (3.3) it thus follows

$$\lim_{\lambda \rightarrow \infty} \psi(\lambda) = \lim_{\Theta \rightarrow 2\pi} \frac{1}{\cos \Theta} = 1. \quad (3.5)$$

Combining the inequalities (3.1) and (3.4)-(3.5) we conclude

$$\lim_{\lambda \rightarrow 0} \psi(\lambda) = \lim_{\lambda \rightarrow \infty} \psi(\lambda) = 1. \quad (3.6)$$

Since the function  $\psi$  is continuous, it admits a global maximum on the positive half line, say

$$\sigma^\dagger \doteq \max_{\lambda > 0} \psi(\lambda).$$

A numerical computation yields

$$\sigma^\dagger = \psi(\lambda^\dagger) = 2.614430844\dots \quad \lambda^\dagger = 0.279949878\dots$$

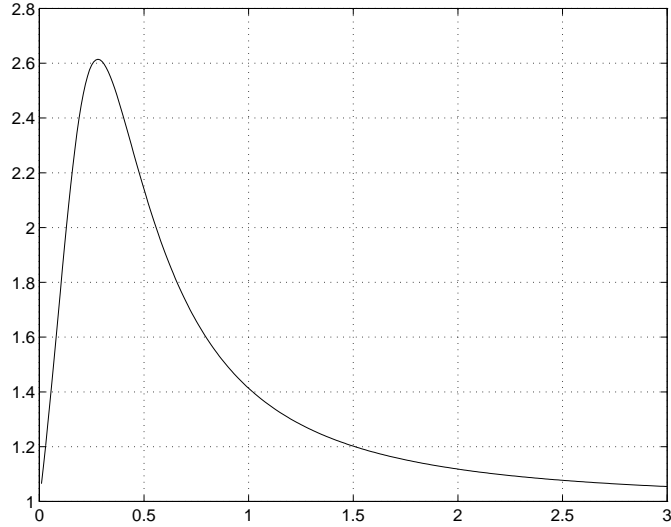


Figure 3. A plot of the function  $\lambda \mapsto \psi(\lambda)$ .

The following result shows that, if the wall construction speed is strictly greater than this maximum, then the single spiral strategy completely encircles the fire within a finite time.

**Theorem 1.** *Let the velocity sets be  $F(x) \equiv B_1$  and assume that  $\sigma > \sigma^\dagger$ . Then for every bounded set  $R_0$  there exists a strategy  $t \mapsto \gamma(t)$  consisting of a single spiral-like wall, which blocks the fire.*

**Proof. 1.** As a preliminary, for any given  $\lambda > 0$  consider the arc of spiral defined by

$$\Gamma_\lambda \doteq \{r = e^{\lambda\theta}; \theta \leq 0\},$$

in polar coordinates. Its total length is measured by

$$\int_{-\infty}^0 \sqrt{r^2(\theta) + \dot{r}^2(\theta)} d\theta = \int_{-\infty}^0 \sqrt{e^{2\lambda\theta} + \lambda^2 e^{2\lambda\theta}} d\theta = \frac{\sqrt{1 + \lambda^2}}{\lambda}.$$

If the wall is constructed at speed  $\sigma$ , the arc  $\Gamma_\lambda$  can be constructed within time

$$\tau_\lambda = \frac{\sqrt{1 + \lambda^2}}{\lambda\sigma}.$$

Let  $R_\lambda$  be the set of points which can be connected to the origin by a curve of length  $\leq \tau_\lambda$ , without ever crossing the spiral  $\Gamma_\lambda$ . As shown in figure 4, part of the boundary of  $R_\lambda$  lies on the curve  $\Gamma_\lambda$ . Another portion of the boundary is made by points  $P$  such that, for some other point  $P' \in \Gamma_\lambda$ , the following conditions hold:

- The segment  $P'P$  is tangent to the spiral  $\Gamma_\lambda$  at  $P'$  and does not touch  $\Gamma_\lambda$  at any other point.
- The total length of the arc of  $\Gamma$  from the origin to  $P'$ , plus the length of the segment  $P'P$  equals  $\tau_\lambda$ .

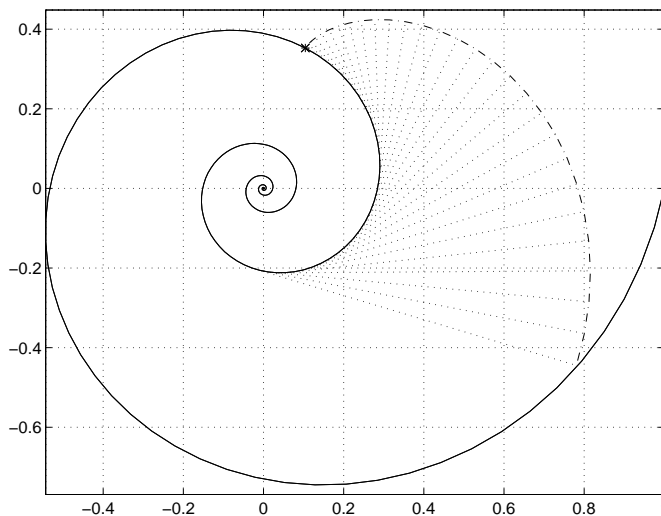


Figure 4. Construction of the set  $R_\lambda$ , in the case  $\lambda = 0.2$ ,  $\sigma \approx 2.7183$ , and  $\tau_\lambda \approx 1.8758$ .

**2.** In view of Lemma 1, it suffices to show that the fire can be blocked if initially it occupies the unit ball  $B_1$ , centered at the origin. In this case, for an initial time interval  $[0, t_1]$  we shall construct a wall along the spiral  $r = e^{\lambda_1 \theta}$ , with

$$\lambda_1 \doteq \frac{1}{\sqrt{\sigma^2 - 1}}.$$

This guarantees that the edge of the fire always remains on one side of the spiral.

For each integer  $k \geq 1$ , call  $t_{1,k}$  the time needed to construct the arc of spiral described in polar coordinates by

$$\Gamma_{1,k} \doteq \{r = e^{\lambda_1 \theta}; 0 \leq \theta \leq 2k\pi\}.$$

Consider the rescaled set

$$\tilde{R}_{1,k} \doteq e^{-2k\pi\lambda_1} R^\gamma(t_{1,k}). \quad (3.7)$$

As  $k \rightarrow \infty$ , we have the convergence of sets  $\tilde{R}_{1,k} \rightarrow R_{\lambda_1}$  in the Hausdorff distance. We now define  $t_1 \doteq t_{1,k_1}$ ,  $\theta_1 = 2k_1\pi$ , for some integer  $k_1$  sufficiently large.

**3.** Consider the point whose cartesian coordinates are

$$P(t_{1,k}) = (e^{2k\pi\lambda_1}, 0),$$

which lies at the edge of the wall, at time  $t_{1,k}$ . Since  $\sigma > \sigma^\dagger$ , for all  $k \geq 2$  this point lies outside the closure of the set  $R^\gamma(t_{1,k})$ . For  $t > t_1$  we can thus choose  $\lambda_2 < \lambda_1$  and construct the wall along a second logarithmic spiral, of the form

$$\Gamma_{2,k} = \{r = e^{\theta_1\lambda_1} \cdot e^{(\theta - \theta_1)\lambda_2}; \theta_1 < \theta < \theta_1 + 2k\pi\}.$$

If  $\lambda_2$  is sufficiently close to  $\lambda_1$ , the fire will always remain in the interior side of this wall. Call  $t_{2,k}$  the time needed to construct the arc of spiral  $\Gamma_{2,k}$ . Letting  $k \rightarrow \infty$ , the rescaled set ,

$$\tilde{R}_{2,k} \doteq e^{-2k\pi\lambda_2} R^\gamma(t_{2,k}) \quad (3.8)$$

approaches  $R_{\lambda_2}$  in the Hausdorff distance. We again observe that, since  $\sigma > \sigma^\dagger$ , for all  $k \geq 2$  the point whose cartesian coordinates are

$$P(t_{2,k}) = (e^{\theta_1 \lambda_1} \cdot e^{2k\pi \lambda_2}, 0)$$

lies outside the closure of the set  $R^\gamma(t_{2,k})$ . We now define  $t_2 \doteq t_1 + t_{2,k_2}$ ,  $\theta_2 = \theta_1 + 2k_2\pi$ , for some integer  $k_2$  large enough.

Continuing this procedure, for  $\theta > \theta_2$  we construct the wall along a tighter logarithmic spiral, of the form  $r = C_3 e^{\lambda_3 \theta}$ , with  $\lambda_3 < \lambda_2$ , etc. . .

**4.** By induction, we thus construct a continuous wall which always confines the fire to one of its sides, whose consecutive portions are arcs of spirals of the form

$$r = C_j e^{\lambda_j \theta} \quad \theta \in [\theta_{j-1}, \theta_j],$$

with  $\lambda_1 > \lambda_2 > \dots$ . A continuity argument based on the rescalings (3.7)-(3.8) shows that one can achieve  $\lambda_{j+1} < \lambda_j - \eta(\lambda_j)$ , for some continuous function  $\lambda \mapsto \eta(\lambda)$ , strictly positive for  $0 < \lambda < \infty$ . Since we are assuming  $\sigma > \sigma^\dagger > 2$ , in a finite number of steps we reach an exponent  $\lambda_m > 0$  small enough so that

$$\frac{\sqrt{1 + \lambda_m^2}}{\lambda_m} > \frac{1}{\sigma} \left( \frac{\sqrt{1 + \lambda_m^2}}{\lambda_m} e^{2\pi \lambda_m} + (e^{2\pi \lambda_m} - 1) \right). \quad (3.9)$$

According to (3.9), the total time taken by the fire to reach the point  $(1, 0)$  starting from the origin and without crossing the spiral

$$\Gamma_m = \{r = e^{\lambda_m \theta}; \quad -\infty < \theta < 2\pi\}$$

is strictly smaller than the time needed to construct both the arc of spiral  $\Gamma_m$  and the segment joining the point  $(x, y) = (e^{2\pi \lambda_m}, 0)$  with the point  $(x', y') = (1, 0)$ .

By a rescaling argument, when the fire is confined on one side of a spiral of the form  $r = C_m e^{\lambda_m \theta}$ , for  $k$  large enough we can then construct the additional segment joining the points

$$(x, y) = (C_m e^{2(k+1)\pi \lambda_m}, 0), \quad (x', y') = (C_m e^{2k\pi \lambda_m}, 0),$$

and completely encircle the fire. □

## 4 - Comparison results for large wall construction speeds

Suppose that the fire is initially burning on the unit circle, and spreads in all directions with unit speed. With our previous notation this means  $R_0 = F(x) = B_1$  for all  $x \in \mathbb{R}^2$ . Let  $\sigma \gg 1$  be the speed at which the wall can be constructed. Aim of this section is to compare different strategies that contain the fire, in the case of very large wall construction speeds, and determine which one yields the smallest burned area.

First, we consider the one-spiral strategy  $\gamma^{(1)}$ . This consists in building one single wall along a spiral-like curve which remains constantly along the edge of the fire.

In addition, for any given integer  $n \geq 1$ , if  $\sigma > 2n$  one can consider the  $2n$ -spiral strategy  $\gamma^{(2n)}$ . This consists in constructing  $2n$  walls along the edge of the fire, each with speed  $\sigma/2n$ . These walls start from  $n$  points equally spaced along the unit circumference. The 2-spiral and the 4-spiral strategies are illustrated in figure 6.

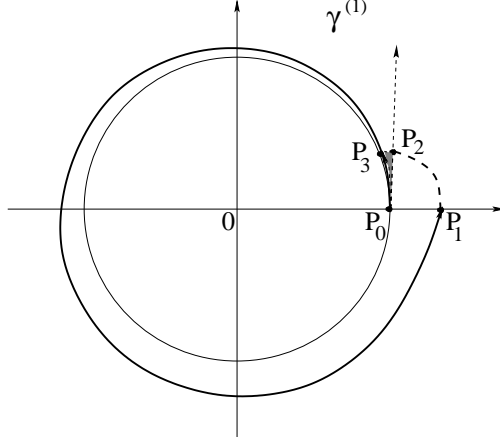


Figure 5. The one-spiral strategy.

In the following, we assume a large construction speed and set

$$\varepsilon \doteq 1/\sigma, \quad \lambda_n \doteq \left( \frac{\sigma^2}{n^2} - 1 \right)^{-1/2} = \frac{n\varepsilon}{\sqrt{1 - n^2\varepsilon^2}},$$

so that

$$\lambda_n = n\varepsilon + \frac{n^3\varepsilon^3}{2} + \mathcal{O}(\varepsilon^5). \quad (4.1)$$

The meaning of the constants  $\lambda_n$  is as follows. Let  $P(t) = (r(t) \cos \theta(t), r(t) \sin \theta(t))$  be the point where one of the walls is constructed, at time  $t$ . Imposing that the construction speed equals  $\sigma/n$  and the point  $P(t)$  lies always on the edge of the burned region, we obtain the identities

$$|\dot{P}|^2 = \dot{r}^2 + r^2\dot{\theta}^2 = (\sigma/n)^2, \quad r(t) = 1 + t \quad (4.2)$$

Assuming that  $\dot{r}$  has constant sign, from (4.2) we deduce that the wall is located along a spiral described by  $r = Ce^{\pm\lambda_n\theta}$ .

The area burned by the fire in the various cases will thus be evaluated as a series of powers of  $\varepsilon$ . This will allow a direct comparison of the different strategies, in the limit  $\varepsilon \rightarrow 0+$ .

If the strategy  $\gamma^{(1)}$  is implemented, the region burned by the fire can be split in three parts (see fig. 4).

- (i) The region  $R_1$  bounded by spiral rotating counter clockwise from  $P_0$  to  $P_1$ , and by the horizontal segment from  $P_1$  back to  $P_0$ .
- (ii) The triangular region  $R_2$  between  $P_0, P_1$  and  $P_2$ .
- (iii) The thin wedge-shaped region  $R_3$  between  $P_0, P_2$  and  $P_3$ .

The first spiral, between  $P_0$  and  $P_1$ , is described in polar coordinates by the equation

$$r = e^{\lambda_1\theta} \quad 0 \leq \theta \leq 2\pi.$$

The area enclosed by this spiral is computed as

$$\text{Area}(R_1) = \frac{1}{2} \int_0^{2\pi} e^{2\lambda_1\theta} d\theta = \frac{e^{4\pi\lambda_1} - 1}{4\lambda_1}. \quad (4.3)$$

Next, the region  $R_2$  is bounded by a second logarithmic spiral which, in a system of polar coordinates centered at the point  $P_0$ , has equation  $r = Ce^{\lambda_1\theta}$ , with  $C = e^{2\pi\lambda_1} - 1$ . Hence

$$\text{Area}(R_2) = \frac{1}{2} \int_0^{\pi/2} (e^{2\pi\lambda_1} - 1)^2 e^{2\lambda_1\theta} d\theta + \mathcal{O}(\varepsilon^3) = \frac{1}{4\lambda_1} (e^{2\pi\lambda_1} - 1)^2 (e^{\pi\lambda_1} - 1) + \mathcal{O}(\varepsilon^3). \quad (4.4)$$

Finally, we observe that  $\text{Area}(R_3) = \mathcal{O}(\varepsilon^3)$ . Recalling (4.1) and expanding the right hand sides of (4.3) and (4.4) in powers of  $\varepsilon$ , we conclude

$$\begin{aligned} \text{Area}(R_1 \cup R_2 \cup R_3) &= \left( \pi + 2\pi^2\varepsilon + \frac{8}{3}\pi^3\varepsilon^2 + \mathcal{O}(\varepsilon^3) \right) + \left( \pi^3\varepsilon^2 + \mathcal{O}(\varepsilon^3) \right) + \mathcal{O}(\varepsilon^3) \\ &= \pi + 2\pi^2\varepsilon + \frac{11}{3}\pi^3\varepsilon^2 + \mathcal{O}(\varepsilon^3). \end{aligned} \quad (4.5)$$

Next, given any integer  $n \geq 1$ , we compute the area burned by the fire if the  $2n$ -spirals strategy  $\gamma^{(2n)}$  is adopted. This area equals  $n$  times the area of a sector  $\Sigma_{2n}$  bounded by the spiral  $r = e^{\lambda_{2n}\theta}$ ,  $\theta \in [0, \pi/n]$ . We now have

$$\text{Area}(\Sigma_{2n}) = \frac{1}{2} \int_0^{\pi/n} e^{2\lambda_{2n}\theta} d\theta = \frac{1}{4\lambda_{2n}} (e^{2\pi\lambda_{2n}/n} - 1). \quad (4.6)$$

For any  $n \geq 1$ , recalling (4.1) and expanding the right hand side in powers of  $\varepsilon$ , we conclude that the area burned by the fire with the strategy  $\gamma^{(2n)}$  is

$$\begin{aligned} 2n \cdot \text{Area}(\Sigma_{2n}) &= \frac{2n}{4\lambda_{2n}} (e^{2\pi\lambda_{2n}/n} - 1) \\ &= \pi + 2\pi^2\varepsilon + \frac{8}{3}\pi^3\varepsilon^2 + \left( 4\pi^2n^2 + \frac{8}{3}\pi^4 \right) \varepsilon^3 + \mathcal{O}(\varepsilon^4). \end{aligned} \quad (4.7)$$

Comparing (4.7) with (4.5) we see that, for  $\varepsilon > 0$  small and hence  $\sigma = 1/\varepsilon$  large, the strategies  $\gamma^{(2n)}$  perform better than  $\gamma^{(1)}$ . The difference essentially accounts for the area of the additional region  $R_2$  burned with the strategy  $\gamma^{(1)}$ . Furthermore, the Taylor expansion (4.7) shows that, up to second order, all strategies  $\gamma^{(2n)}$  are equivalent. However, the analysis of third order terms shows that  $\gamma^{(2)}$  performs better than  $\gamma^{(4)}$ , which in turn is better than  $\gamma^{(6)}$ , and so on.

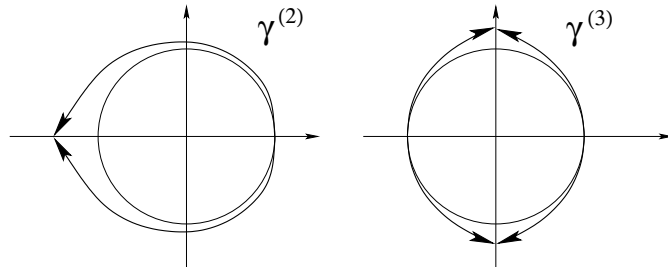


Figure 6. The two-spirals and four-spirals strategies.

## 5 - The non-isotropic case

We now consider the case where the fire propagates with different velocities in different directions. For a given set  $F \subset \mathbb{R}^2$  of fire propagation speeds, our aim is to find for which wall construction speeds  $\sigma$  one can entirely encircle the fire. Under the assumptions (1.10)-(1.11), the following analysis shows that the wall speed should be strictly larger than the vertical width of the velocity set  $F$ .

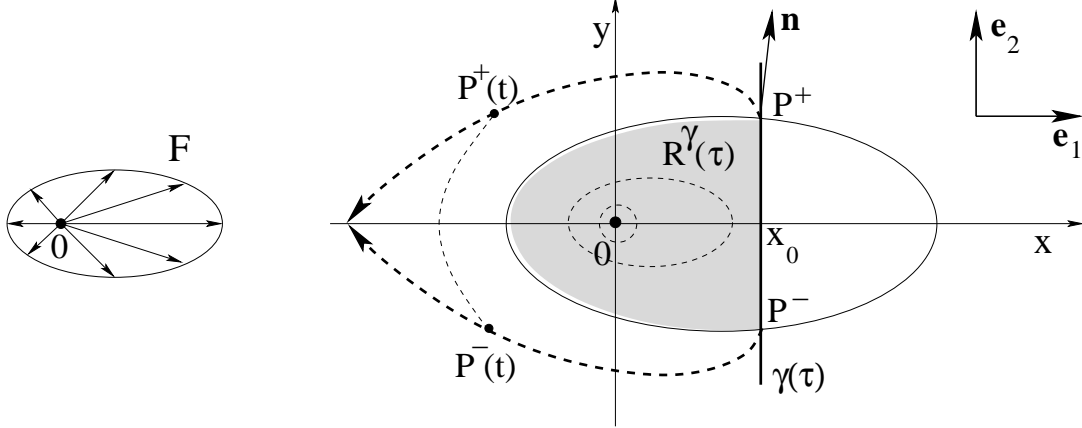


Figure 7. Construction of a wall encircling the fire, in the non-isotropic case.

**Theorem 2.** *Assume that the set  $F$  of fire propagation speeds satisfies (1.10)-(1.11). If*

$$\frac{\sigma}{2} > \max_{\theta \in [0, \pi]} \rho(\theta) \sin \theta, \quad (5.1)$$

*then, for every bounded initial set  $R_0$ , there exists a strategy  $\gamma(\cdot)$  that completely encircles the fire within finite time.*

**Proof.** Assume  $\sigma/2 > \mu$ , where

$$\mu \doteq \max_{\theta \in [0, \pi]} \rho(\theta) \sin \theta = \max_{\mathbf{v} \in F} \mathbf{e}_2 \cdot \mathbf{v}. \quad (5.2)$$

By a translation in space, and by replacing  $R_0$  with a larger set, we can assume that the initial set  $R_0$  coincides with the ball  $B_R$  centered at the origin with radius  $R$ . Moreover, by possibly replacing the set  $F$  with its  $\varepsilon$ -neighborhood

$$B(F, \varepsilon) \doteq \{ \mathbf{v} + \mathbf{w}; \mathbf{v} \in F, |\mathbf{w}| \leq \varepsilon \},$$

we can also assume that the set  $F$  has a  $\mathcal{C}^1$  boundary, and that the map  $\theta \mapsto \rho(\theta)$  in (1.10) is continuously differentiable. The proof will be achieved in two steps.

1. On an initial interval of time  $[0, \tau]$ , we construct a vertical wall:

$$\gamma(t) = \left\{ (x_0, y); y \in \left[ -\frac{\sigma t}{2}, \frac{\sigma t}{2} \right] \right\}.$$

We claim that, if  $x_0 > 0$  is chosen sufficiently large, then the fire will never reach the points  $(x_0, \pm\sigma t/2)$  at the edge of the walls. Indeed, there exists  $t^*$  sufficiently large so that

$$(1 + t\rho(\theta)) \sin \theta \leq 1 + t\rho(\pi/2) < t\sigma/2$$

for all  $t \geq t^*$ ,  $\theta \in [0, \pi/2]$ . On the other hand, choosing  $x_0$  large enough, we can guarantee that

$$(1 + t\rho(\theta)) \cos \theta \leq (1 + t\rho(\pi/2)) < x_0$$

for all  $t \leq t^*$ . Combining the above inequalities we conclude that the system

$$\begin{cases} (1 + \tau\rho(\theta)) \cos \theta = x_0, \\ (1 + \tau\rho(\theta)) \sin \theta = \tau\sigma/2, \end{cases}$$

has no solution for  $t \geq 0$ ,  $\theta \in [0, \pi/2]$ . This proves our claim.

As a consequence, the reachable set remains on the left side of the wall namely

$$R^\gamma(t) = (B_R + tF) \cap \{(x, y); x < x_0\}.$$

where  $B_R + tF = \{\mathbf{v}_1 + t\mathbf{v}_2; |\mathbf{v}_1| \leq R, \mathbf{v}_2 \in F\}$ .

**2.** Let  $x_0 > 0$  be chosen as in the previous step. For  $\tau > 0$  large, let  $P^\pm$  be the points on the boundary of the set  $B_R + tF$  which lie on the vertical line where  $x = x_0$  (see figure 7). Choosing  $\tau$  sufficiently large, we can assume that the outer normal  $\mathbf{n}$  to the set  $B_R + \tau F$  at the point  $P^+$  is arbitrarily close to the unit vector  $\mathbf{e}_2 = (0, 1)$ . Therefore, from (5.2) it follows

$$\max_{\mathbf{v} \in F} \mathbf{v} \cdot \mathbf{n} < \frac{\sigma}{2}. \quad (5.3)$$

For  $t > \tau$ , a simple blocking strategy consists in building two boundary walls on the edge of the fire. As shown in fig. 7, the moving edges of the walls will be the two points  $P^+(t), P^-(t)$ . These are determined by the two equations

$$P^\pm(t) \in \partial(B_R + tF), \quad (5.4)$$

$$|\dot{P}^+| = |\dot{P}^-| = \frac{\sigma}{2}. \quad (5.5)$$

By assumption, in a neighborhood of the points  $P^\pm(t)$ , the normal velocity at which the sets  $R(t) \doteq (B_R + tF)$  increase is strictly smaller than  $\sigma/2$ . Therefore, the two conditions (5.4)-(5.5) yield a well defined O.D.E., describing how the points  $P^+, P^-$  move in time. Using polar coordinates, we have

$$P^\pm(t) = (\pm r(t) \sin \theta(t), r(t) \cos \theta(t)).$$

For any angle  $\theta$ , define the unit normal  $\mathbf{n}(\theta) = (\cos \theta, \sin \theta)$ . Notice that the assumption (1.11) implies that the map

$$\theta \mapsto \sup_{\mathbf{v} \in F} \mathbf{n}(\theta) \cdot \mathbf{v} \quad \theta \in [0, \pi]$$

is non-increasing. As a consequence, the map  $t \mapsto \dot{\theta}(t)/\dot{r}(t)$  is strictly positive and non-decreasing. Therefore, for  $t \geq \tau$  the walls remain inside the region bounded by the two logarithmic spirals

$$\frac{d\theta(r)}{dr} = \pm \frac{\dot{\theta}(\tau)}{\dot{r}(\tau)}$$

starting at the points  $P^\pm(\tau)$ . Within finite time we have  $\theta(T) = \pi$ , and the fire is entirely surrounded by the wall, constructed along the vertical segment  $P^-P^+$  and the two spirals.  $\square$

## 6 - Numerical results

In this final section we present the result of two numerical simulations, comparing different strategies, first in the isotropic, then in the non-isotropic case.

As a preliminary, we observe that the location of boundary arcs can be described by an O.D.E., as follows. For  $P = (x, y) \in \mathbb{R}^2$ , call  $T(P)$  the minimum time needed by the fire to reach the point  $P$ , namely

$$T(P) \doteq \inf \left\{ t \geq 0; \quad P \in R(t) \right\}.$$

For the general theory of Hamilton-Jacobi equations which determine the minimal time function, we refer to [2]. If a wall is constructed with speed  $\sigma$  along the boundary of the reachable set, then the construction point  $P(t)$  must satisfy  $T(P(t)) = t$ . From the identities

$$\nabla T(P) \cdot \dot{P} \equiv 1, \quad |\dot{P}| = s, \quad (6.1)$$

calling  $\theta$  the angle between the vectors  $\nabla T$  and  $\dot{P}$ , one obtains

$$\cos \theta = \frac{1}{|\nabla T(P)| \cdot s}. \quad (6.2)$$

In the isotropic case, with  $R_0 = B_1 = F$ , in absence of walls the time function is simply

$$T(x, y) = \begin{cases} 0 & \text{if } x^2 + y^2 \leq 1, \\ \sqrt{x^2 + y^2} - 1 & \text{if } x^2 + y^2 > 1. \end{cases} \quad (6.3)$$

Somewhat more generally, assume that  $R_0 = B_1$  but let  $F$  be the ice cream cone

$$F \doteq \left\{ \lambda \mathbf{v}; \quad |\mathbf{v} - \kappa \mathbf{e}_1| \leq 1, \quad \lambda \in [0, 1] \right\}, \quad (6.4)$$

as shown in fig. 1, left. Here  $\mathbf{e}_1 = (1, 0)$  is a unit vector in  $\mathbb{R}^2$ . In absence of walls, the reachable set at time  $t$  is computed as

$$R(t) = R_0 + tF = \text{co}(B_1 \cup B(t\kappa \mathbf{e}_1, t + 1)),$$

where  $\text{co}(S)$  denotes the convex closure of a set  $S$ . On the region where the boundary of  $R(t)$  coincides with the boundary of the ball centered at  $t\kappa \mathbf{e}_1$  with radius  $t + 1$ , we thus have

$$(x - \kappa t)^2 + y^2 = (t + 1)^2.$$

When  $\kappa \geq 0$ ,  $\kappa \neq 1$ , the minimum time function is

$$T(x, y) = \frac{(\kappa x + 1) - \sqrt{(\kappa x + 1)^2 - (\kappa^2 - 1)(x^2 + y^2 - 1)}}{\kappa^2 - 1}, \quad (6.5)$$

restricted to the region where the argument of the square root is non-negative. In the special case where  $\kappa = 1$ , (6.5) should be replaced by

$$T(x, y) = \frac{x^2 + y^2 - 1}{2(x + 1)} \quad (x > -1). \quad (6.6)$$

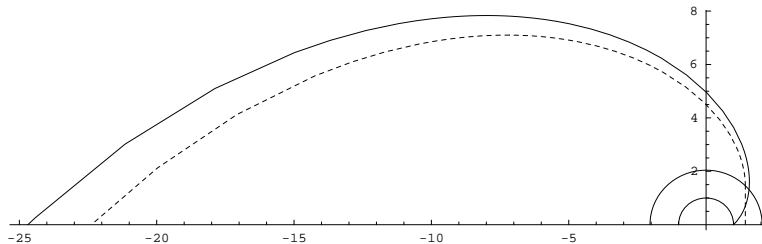


Figure 8. The wall constructed by a spiral, and circle+spiral strategies.

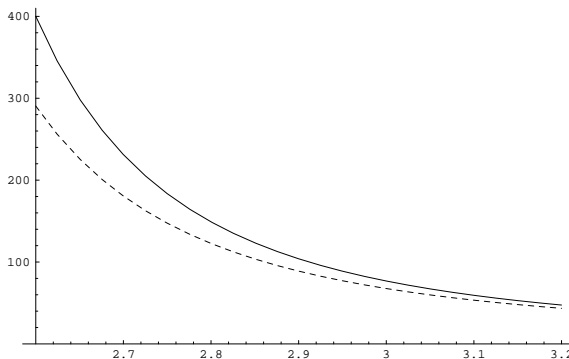


Figure 9. Areas of the burned regions, using different strategies.

Since we shall consider strategies which are symmetric w.r.t. the  $x$ -axis, it is convenient to call  $2\sigma$  the construction speed. Figures represent the part of the walls on the half plane where  $y > 0$ .

Our first numerical simulation compares the burned regions adopting two strategies, both symmetric w.r.t. the  $x$ -axis:

- (i) Two logarithmic spirals (solid lines) are constructed along the edge of the fire, each with speed  $\sigma$ .

In this case, the computation is explicit. In polar coordinates, the upper spiral is described by

$$r = e^{\lambda\theta} \quad \lambda = (\sigma^2 - 1)^{-1/2}, \quad \theta \in [0, \pi].$$

- (ii) An arc of a circle is first constructed away from the fire. Then, at the time where the edge of the fire reaches the construction points, the wall is continued in the form of two spirals (dashed lines).

By standard necessary conditions in the Calculus of Variations, it is known that the portion of walls constructed away from the fire must be an arc of a circle [3, 5]. By the further conditions proved in [3], at the points of juncture the circle and the spiral walls must be tangent. For each time  $\tau$ , we determine a unique arc of circle which has the center on the  $x$ -axis, has length  $\sigma\tau$ , has extreme points  $P_\tau^+$ ,  $P_\tau^-$  on the circumference  $x^2 + y^2 = (1 + \tau)^2$ , and is tangent to the spiral of polar equation  $r = c_\sigma \cdot e^{\lambda\theta}$ , at the point  $P_\tau^+$ . This yields a one-dimensional family of possible arcs, depending on the time  $\tau$  at which the junction occurs. Optimizing over the choice of  $\tau$ , we then determine the strategy minimizing the total burned area.

Figure 8 shows the walls constructed by the two strategies, in the case  $\sigma = 1.4$ . Notice that here the circular portion of the wall is very short, and indeed it looks almost like a straight line.

For construction speeds  $2\sigma \in [2.6, 3.2]$ , a comparison of the areas of the burned regions is given in Figure 9.

In the last numerical simulation we consider the non-isotropic case, where  $F$  is given by (6.4) with  $\kappa = 3$  while the construction speed for each half of the wall is  $\sigma = 4.1$ . Initially, the fire covers the unit disc. Figure 10 shows two different burned regions. One corresponds to wall construction always along the edge of the fire (solid line). The second corresponds to the optimal strategy (dashed line), obtained by first constructing an arc of circle away from the fire, then two arcs (always symmetric w.r.t. the  $x$ -axis) along the fire boundary. As before, the point of junction is chosen in order to achieve tangency of the two curves, and minimize the total burned area.

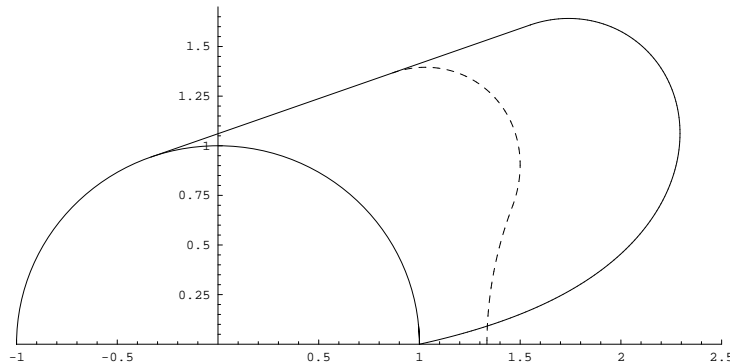


Figure 10. Area burned using two different strategies, in the non-isotropic case.

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