On Hydrodynamics of Viscoelastic Fluids

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Abstract

In this paper, we will study a hydrodynamic system describing fluids with viscoelastic properties. After a brief examination on the relations between several models,we shall concentrate on a few analytical issues concerning them. In particular, we establish local existence and global existence (with small initial data) of classical solutions for an Oldroyd- system without artificially postulated damping mechanism.

1 Introduction

Viscoelastic material can be viewed as the intermediate states between the fluids and solids. The material exhibits elastic behavior, such as the memory effects, as well as the fluid properties. Many complicated hydrodynamic and rheological behavior of complex fluids can be regarded as consequences of the internal elastic properties. Often these microscopic or molecular elastic properties can also be attributed to heterogeneity and the electro-magnetic behavior of the materials [19, 26, 1, 3]. It is the interaction between these (microscopic) elastic properties and the (macroscopic) fluid motions that gives not only the rich and complicated rheological phenomena, but also formidable challenges in analysis and numerical simulations.

In this first one of a series of papers on the subject, we will make a preliminary mathematical study on these systems. In fact, we shall concentrate on a simple hydrodynamic system for viscoelastic fluids that it can be also viewed as an Ericksen-Leslie system (for the evolutions of liquid crystals or polymeric liquids) without certain viscous and dissipative effect in parts of the equations.

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1.1 Flow map and deformation tensor

In the context of hydrodynamics, the basic variable is the flow map (particle trajectory) $x(X,t)$. X is the original labeling (Lagrangian coordinate) of the particle. It is also referred to as material coordinate. x is the current (Eulerian) coordinate and referred to as reference coordinate. For a given velocity field $v(x,t)$ the flow map is defined by the following ordinary differential equation:

$$x_t(X,t) = v(x(X,t),t), \quad x(X,0) = X. \tag{1.1}$$

The deformation tensor $F(X,t)$ is defined as

$$F(X,t) = \frac{\partial x}{\partial X}. \tag{1.2}$$

When look in the Eulerian coordinate, we can define $\tilde{F}(x,t)$ such that $\tilde{F}(x(X,t),t) = F(X,t)$. With no ambiguity, we will not distinguish these two notations in this paper. Applying the chain rule, we see that $F(x,t)$ satisfies the following transport equation [21, 11, 19]:

$$F_t + v \cdot \nabla F = \nabla v F, \tag{1.3}$$

which stands for ${F_{ij}}_t + v_k \nabla_k F_{ij} = \nabla_k v_i F_{kj}$. Here we point out that in this paper, we use the notation $F_{ij} = \frac{\partial x_i}{\partial X_j}$ and $(\nabla v)_{ij} = \frac{\partial v_i}{\partial x_j}$. This is different from notations in other papers by a transpose, for instance [19].

The incompressibility is represented as

$$\det F = 1. \tag{1.4}$$

By the identity of the variation of the determinant of a tensor

$$\delta \det F = \det F \operatorname{tr}(F^{-1}\delta F), \tag{1.5}$$

we see that $\nabla \cdot v = 0$. Moreover, we assume that the density $\rho = \rho_0$ to be a constant. This will replace the equation for the conservation of mass :

$$\rho_t + \nabla \cdot (\rho v) = 0. \tag{1.6}$$

Let us denote $(\nabla \cdot F)_j = (\nabla_i F_{ij})$, then we have [21, 11, 19]

$$(\nabla_i F_{ij})_t + v_k \nabla_k (\nabla_i F_{ij}) + \nabla_i v_k (\nabla_k F_{ij}) = \nabla_k v_i (\nabla_i F_{kj}) + \nabla_i \nabla_k v_i F_{kj}.$$

Using the incompressibility and switch the indices i and k of the first term on the right hand side, we have:

$$(\nabla \cdot F)_t + v \cdot \nabla(\nabla \cdot F) = 0. \tag{1.7}$$

To describe the elastic properties of the fluid, we associate the Lagrangian for the fluid motions with an elastic energy functional for some internal variables. In this case, the internal variable is the deformation gradient of the fluid, $F(x,t)$. Though our discussions below will be for a special elastic energy functional of the Hookean linear elasticity, it does not reduce the essential difficulties for analysis. Indeed, all the results we described here can be easily generalized to a more general class of elastic energy functionals for the deformation tensor $F(x,t)$ (see [21] for some related discussions).

1.2 Force Balance and Oldroyd-B systems

For a general viscoelastic fluid, we start from the following equation for the conservation of momentum :

$$\rho(v_t + v \cdot \nabla v) = \nabla \cdot \tau, \tag{1.8}$$

where τ is total stress. In a Newtonian flow, one assume the constitutive relation $\tau = -pI + \mu D$, where p is the pressure, μ the viscosity and $D = \frac{\nabla v + \nabla^T v}{2}$ is the strain rate.

There have been many attempts to capture different phenomena for non-Newtonian fluids, such as those of Ericksen-Rivlin [29, 27] or for high-grade fluid [15], Ladyzhenskaya [18] where τ is nonlinear in the strain rate D, and by Necas's group where viscosity depending on both D and p [13, 24]. All these models involve only instantaneous constitutive relation between the stress and strain.

For the nonlocal (in time) constitutive equations, there are the Maxwell model $\tau_t + \gamma\tau = \mu D$, the transport model $\tau_t + v \cdot \nabla\tau + \gamma\tau = \mu D$, and the Oldroyd (upper convective) model

$$\tau_t + v \cdot \nabla\tau - \nabla v\tau - \tau\nabla v^T + \gamma\tau = \mu D, \tag{1.9}$$

The constant γ in the above models represents the time scale for the elastic relaxation. It is associate to the Debra number $De = \frac{\mu}{\gamma}$, which indicates the relation between the characteristic flow time and the characteristic elastic time scales [1].

We note that, there are other types of Oldroyd models. Some are associated with the different ways in which the stress tensor is transported. For instance, the Johnson-Segaman model is just a linear combination of the upper convective and the lower convective Oldroyd models.

We can also look at the following modified Oldroyd model:

$$\tau = -pI + \mu D + \tau_1, \tag{1.10}$$

and the elastic stress τ_1 satisfies the transport equation:

$$\tau_{1t} + v \cdot \nabla\tau_1 - \nabla v\tau_1 - \tau_1\nabla v^T + \gamma\tau_1 = \delta I, \tag{1.11}$$

The equation (1.11) is related to the modified Oldroyd model (1.9) by a simply change of variable as $\tau_1 = \tau - \eta I$, where $\eta = \mu/2$ [25].

Denote by $C = FF^T$, the tensor is usually called the Cauchy-Green strain tensor and $B = C^{-1}$ is the finger tensor [19, 11, 25]. In particular, the equation (1.11) is equivalent to

$$(F^{-1}\tau_1F^{-T})_t + v \cdot \nabla(F^{-1}\tau_1F^{-T}) = -\gamma(F^{-1}\tau_1F^{-T}) + \delta F^{-1}F^{-T},$$

Hence, we can implicitly write the solution in the form :

$$\begin{aligned} \tau_1(x,t) &= \exp\{-\gamma t\}F(x,t)\tau_1(x,0)F^T(x,t) \\ &\quad +\delta\int_{-\infty}^{t} \exp\{-\gamma(t-s)\}F(x,t)F^{-1}(x,s)F^{-T}(x,s)F^T(x,t)\,ds. \end{aligned} \tag{1.12}$$

From this, it is obvious that τ_1 is positive definite. In fact, in this case, we can define the *induced deformation* tensor $F_1 = \sqrt{\tau_1}$.

The following fact is obvious.

Lemma 1.1 *If a tensor* τ *satisfies the equation:*

$$\tau_t + v \cdot \nabla\tau - \nabla v\tau - \tau\nabla v^T = 0, \tag{1.13}$$

and the initial condition $\tau(x, 0) = \tau_0(x)$ *is positive definite, then*

$$\tau(x, t) = F\tau_0 F^T. \tag{1.14}$$

Moreover, the induced deformation tensor $F_1 = \sqrt{\tau}$ *satisfies the same equation as (1.3):*

$$(F)_t + v \cdot \nabla F = \nabla v F.$$

We remark that this lemma, together with the results in [21] will allow us to obtain global (Larey) weak solutions for the case the induced strain of viscoelasticity is small. We notice that this type of results are different from the existence results of [9, 10, 26]. It is based on arguments in [21] and the following energy law for the Oldroyd model:

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho|v|^2 + \frac{1}{2}\mathrm{tr}\,\tau_1\,dx = -\int_\Omega \mu|D|^2. \tag{1.15}$$

We shall outline in the next subsection the basic approach for existence of weak solutions of [21]. However the main goal of this paper is to establish the global existence of classical solutions. Recently, there are some mathematical results concerning these systems, see for examples [20, 2, 5]. However, even for the global existence of small (in a suitable sense) weak solutions, our results here are stronger because we do not add additional damping mechanisms or (viscous) dissipative effects.

1.3 Energetic Variational Formulation

As in [21], in order to study the mixture of a fluid with a visco-elastic solid, one writes the momentum equation for the viscoelastic materials in the Eulerian framework. Assuming the elastic energy of the material is $W(F)$ where $F = [\partial x/\partial X]$ is the deformation tensor (strain). The following system (in weak form) gives the force balance equations (linear momentum equations):

$$\int_\Omega [\rho(v_t + (v \cdot \nabla)v) \cdot u - p\nabla \cdot u + \tau \cdot \nabla u\,dx = \int_\Omega \rho f \cdot u dx, \tag{1.16}$$

for any test function u, the elastic stress: $\tau = \mu D(v) + (1/J)S(F)F^T$, where $S(F) = [\partial W/\partial F]$ takes the Piola Kirchhoff form. Here we also adopt the constraint $J = det(F) = 1$ for incompressibility. This momentum equation can be derived through the least action principle (Hamilton's principle). The action functional take the form:

$$A(x) = \int_0^T \int_{\Omega_0} \frac{1}{2}\rho|x_t(X, t)|^2 - W(F)\,dXdt, \tag{1.17}$$

where Ω_0 is the original domain occupied by the material. We use the fact that $J = \det F = 1$.

Let us take any one-parameter family of volume preserving flow map $x^\epsilon(X,t)$ with $\frac{dx^\epsilon}{d\epsilon}|_{\epsilon=0} = y$. From the fact that $J = \det F = 1$ and the identity (1.5), we have that $\nabla\cdot y = 0$. Now the equation (1.16) (without the viscous dissipation term) can be viewed as the following first variation of A with respect to the flow map (or the deformation) x:

$$\frac{d}{d\epsilon} A(x^\epsilon)|_{\epsilon=0} = 0. \tag{1.18}$$

Conventionally, one studies the elasticity through the force balance equations, using the Lagrangian coordinate. For the problems here we use the trajectory $x(X,t)$ as the unknown variable (or the displacement $x - X$). The equation reads as

$$\rho x_{tt} = -\frac{\delta W}{\delta x} = \nabla_X \cdot W_F + \nabla_X \cdot (pF^{-1}), \tag{1.19}$$

where p is the Lagrangian multiplier due to the incompressibility condition, and it satisfies the energy law:

$$\frac{d}{dt}\int_{\Omega_0} \frac{1}{2}\rho|x_t|^2 + W(F)\,dX = 0, \tag{1.20}$$

noticing the relation that $F^{-1} : F_t = \mathrm{tr}(F^{-1}F_t) = \frac{1}{\det F}\frac{d}{dt}\det F = 0$. In the case of Hookean (linear) elasticity, $W(F) = |F|^2 = \mathrm{tr}\,(FF^T)$, it becomes the usual wave equation:

$$\rho x_{tt} = \nabla_X \cdot W_F = \nabla_X \cdot F + \nabla_X \cdot (pF^{-1}) = \Delta_X x + \nabla_X \cdot (pF^{-1}). \tag{1.21}$$

A closely related problem to the above system was recently studied by Sideris-Thomases [28], see also the last two sections of the present paper for a more detailed discussion.

We point out that it will be difficult to input the frame indifferent viscous term in the above equations.

For the system (1.16), one can easily check that it satisfies the energy law (second law of thermodynamics [11]):

$$\frac{d}{dt}\int_{\Omega} \frac{1}{2}\rho|v|^2 + W(F)\,dx = -\int_{\Omega} \mu|D|^2\,dx. \tag{1.22}$$

We note that even in the case of linear elasticity for the (microscopic)internal variable $F(x,t)$, the (macroscopic) elastic stress term $\tau_2 = W_F F^T = FF^T$ is still nonlinear. In fact, it is always of the same order of the nonlinearity as the elastic energy density. This is probably the main analytical difficulty for these Oldroyd type systems. On the other hand, we can make the following observation. Using the fact that F satisfies the transport equation (1.3), we have

$$\tau_{2t} + v\cdot\nabla\tau_2 - \nabla v\tau_2 - \tau_2\nabla v^T = 0 \tag{1.23}$$

and we can recover the Oldroyd system (without the damping). Notice in this case that $W(F) = \operatorname{tr}\tau_2$. Hence the two energy laws are also consistent with each other.

In fact, we can start with the above energy law and derive the linear momentum equations (hence the constitutive equations). This is also the approach that was used by Ericksen in the study of liquid crystal materials [6] and Gurtin for phase transitions [7].

We finally note that the linear transport equation $F_t + (v\cdot\nabla)F = \nabla v\, F$ for the tensor case may not be treated directly in the framework of [4] or [16]. Nonetheless, one may apply the div-curl lemma [30] to obtain weak solutions [21] in some cases. In general, however, it is rather difficult to achieve the compactness of sequences of solutions, see [20]. The method used in [21] was to consider the polar decomposition. Let R be the rotation part and let the symmetric part U be small. In these new variables, one has the equations $R_t + u\cdot\nabla R = W(v)R$, $U_t + v\cdot\nabla U = R^T D(v)R$, where $F = R(I+U)$, $D(v)$, $W(v)$ are the symmetric skew components of ∇v. This is not the usual linear elastic formulation, rather, it is in the same sitting as that in the famous work of F. John [14]. If we linearize the elastic stress

$$D\mathcal{W}(F)F^T = R(D\mathcal{W}(I) + D\mathcal{W}(I)U + \mathcal{C}(U) + O(U^2))R^T. \tag{1.24}$$

Here $\mathcal{C}(U)_{j\beta} = D^2\mathcal{W}(I)(U)_{j\beta} = \frac{\partial^2\mathcal{W}}{\partial F_{i\alpha}\partial F_{j\beta}}(I)U_{i\alpha}$. The special form of the equation of R allowed one to get an approximate system for R and to generalize the tools for scalar transport equations [4] to this small strain case, and eventually leaded to the global existence of the approximate system [21].

The paper is written as follows. In section 2.1, we shall discuss various equivalent formulations of the Oldroyd models,and some related evolution systems. Then the rest of the paper will simply devoted to a study of a particular system which is closely related also to the Ericksen-Leslie system, see [22]. This particular system is equivalent to the Oldroyd model in the 2-D case. It turns out many analysis on this particular model can be easily generalized to the Oldroyd models as well, we shall simply add, at various places of the paper, remarks to indicate how that can be done. To simply the presentations, we consider only the Cauchy problems on either the entire space or on a periodic domain. The problems on a bounded smooth domain with the Dirichlet conditions are treated in our forthcoming paper. In the section 2.2, we established local existence and uniqueness of smooth solutions. We also characterize when a local smooth solution may blow up in a finite time. In section 2.3, we establish global existence of smooth and small solutions. Such a result can naturally be used to establish global existence of large solutions when the viscosity is sufficiently large. The last two sections are devoted to the inviscid case. These related problems in the inviscid case was recently studied by Sideris-Thomases also, see [28]. They used dispersive estimates for certain linear wave equations to obtain global existence of small (in certain weighted Sobolev spaces) solutions. Here we observed first that our system is, in fact, a non-standard symmetric hyperbolic system that also takes into account of the incompressibility of the fluids. This enables we to establish the local existence of smooth solutions. One can also establish directly the global existence of small solutions in the inviscid case as that in [28] by using similar type of the dispersive estimates as well as our formulations.

2 Existence of Solutions

2.1 Preliminaries

We shall study the basic existence and uniqueness questions for the Cauchy problem of the Oldroyd systems. Let the spatial domain Ω be either a bounded domain in $\mathbf{R}^2$ (or $\mathbf{R}^3$) with smooth boundary,or the entire plane (space), or a periodic box. The (linear) viscoelastic fluid system of the Oldroyd model takes the following form:

$$\begin{cases} F_t + v\cdot\nabla F = \nabla v F, \\ v_t + v\cdot\nabla v + \nabla p = \mu\Delta v + \nabla\cdot(FF^T), \\ \nabla\cdot v = 0, \end{cases} \tag{2.1}$$

where the $i-$th component of $\nabla\cdot(FF^T)$ on the right hand side of the momentum equation is $\nabla_j(F_{ik}F_{jk})$. The initial conditions for unknowns are:

$$F(x,0) = F_0(x),\; v(x,0) = v_0(x), \tag{2.2}$$

with assumption $\det F_0 = 1$. In the case of bounded domains, we choose the following Dirichlet boundary condition: for any x on the boundary $\partial\Omega$,

$$v(x,t) = 0. \tag{2.3}$$

With this condition, there is no need F to be prescribed on the boundary. As noted before, the system has formally the basic energy identity :

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|v|^2 + \frac{1}{2}|F|^2\,dx = -\int_\Omega \mu|\nabla v|^2\,dx. \tag{2.4}$$

From the identity $(\nabla\cdot F)_t + v\cdot\nabla(\nabla\cdot F) = 0$, and if we assume that $\nabla\cdot F_0 = 0$,then for all latter time we have that $\nabla\cdot F = 0$. Thus $F = \nabla\times\phi$ where ϕ is a matrix. In the 2-dimensional case,

$$F = \begin{pmatrix} -\partial_2\phi_1 & -\partial_2\phi_2 \\ \partial_1\phi_1 & \partial_1\phi_2 \end{pmatrix}, \tag{2.5}$$

and

$$\begin{aligned} FF^T &= \begin{pmatrix} -\partial_2\phi_1 & -\partial_2\phi_2 \\ \partial_1\phi_1 & \partial_1\phi_2 \end{pmatrix}\begin{pmatrix} -\partial_2\phi_1 & \partial_1\phi_1 \\ -\partial_2\phi_2 & \partial_1\phi_2 \end{pmatrix} \\ &= \begin{pmatrix} (\partial_2\phi_1)^2 + (\partial_2\phi_2)^2 & -\partial_1\phi_1\partial_2\phi_1 - \partial_2\phi_2\partial_1\phi_2 \\ -\partial_1\phi_1\partial_2\phi_1 - \partial_2\phi_2\partial_1\phi_2 & (\partial_1\phi_1)^2 + (\partial_1\phi_2)^2 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}\nabla\cdot(FF^T) &= \begin{pmatrix} \partial_1((\partial_2\phi_1)^2+(\partial_2\phi_2)^2)-\partial_2(\partial_1\phi_1\partial_2\phi_1+\partial_2\phi_2\partial_1\phi_2) \\ -\partial_1(\partial_1\phi_1\partial_2\phi_1+\partial_2\phi_2\partial_1\phi_2)+\partial_2((\partial_1\phi_1)^2+(\partial_1\phi_2)^2) \end{pmatrix} \\ &= \nabla((\partial_2\phi_1)^2+(\partial_2\phi_2)^2+(\partial_1\phi_1)^2+(\partial_1\phi_2)^2) \\ &\quad - \begin{pmatrix} \partial_1((\partial_1\phi_1)^2+(\partial_1\phi_2)^2)+\partial_2(\partial_1\phi_1\partial_2\phi_1+\partial_2\phi_2\partial_1\phi_2) \\ \partial_1(\partial_1\phi_1\partial_2\phi_1+\partial_2\phi_2\partial_1\phi_2)+\partial_2((\partial_2\phi_1)^2+(\partial_2\phi_2)^2) \end{pmatrix}.\end{aligned}$$

Moreover

$$\begin{aligned}&\partial_1((\partial_1\phi_1)^2+(\partial_1\phi_2)^2)+\partial_2(\partial_1\phi_1\partial_2\phi_1+\partial_2\phi_2\partial_1\phi_2) \\ &= \Delta\phi_1\partial_1\phi_1+\Delta\phi_2\partial_1\phi_2+\frac{1}{2}\partial_1(|\partial_1\phi_1|^2+|\partial_1\phi_2|^2+|\partial_2\phi_1|^2+|\partial_2\phi_2|^2),\end{aligned}$$

and

$$\begin{aligned}&\partial_1(\partial_1\phi_1\partial_2\phi_1+\partial_2\phi_2\partial_1\phi_2)+\partial_2((\partial_2\phi_1)^2+(\partial_2\phi_2)^2) \\ &= \Delta\phi_1\partial_2\phi_1+\Delta\phi_2\partial_2\phi_2+\frac{1}{2}\partial_2(|\partial_1\phi_1|^2+|\partial_1\phi_2|^2+|\partial_2\phi_1|^2+|\partial_2\phi_2|^2).\end{aligned}$$

If we denote $\phi=(\phi_1,\phi_2)$, we see that ϕ is a volume preserving map with $\det(\nabla\phi)=1$. The Oldroyd system can also be transformed (after adjusting the order and sign) into:

$$\begin{cases} \phi_t+v\cdot\nabla\phi=0, \\ v_t+v\cdot\nabla v+\nabla p=\mu\Delta v-\sum_{i=1}^2\Delta\phi_i\nabla\phi_i, \\ \nabla\cdot v=0. \end{cases} \tag{2.6}$$

The initial conditions become:

$$\phi(x,0)=\phi_0,\ v(x,0)=v_0(x). \tag{2.7}$$

Here we assume that the boundary condition ϕ_0 satisfy $\det(\nabla\phi_0)=1$ (in this case, from the transport equation of ϕ, we see that $\det(\nabla\phi)=1$ at all time. In the case of bounded domains, the boundary conditions take the form: for any x on the boundary $\partial\Omega$,

$$v(x,t)=0. \tag{2.8}$$

Again, there is no need to prescribe ϕ on the boundary. Finally the basic energy law becomes:

$$\frac{d}{dt}\int_\Omega\frac{1}{2}|v|^2+\frac{1}{2}|\nabla\phi|^2\,dx=-\int_\Omega\mu|\nabla v|^2\,dx. \tag{2.9}$$

For convenience,from here on we will denote $\sum_{i=1}^2\Delta\phi_i\nabla\phi_i$ by $\Delta\phi\nabla\phi$.

Remark. The induced stress $\Delta\phi\nabla\phi$ can also be derived from the part of the action functional representing the elasticity of the fluids:

$$\int_0^T\int_{\Omega_0}W(F)\,dXdt=\int_0^T\int_{\Omega_0}\frac{1}{2}|\nabla^\perp\phi|^2\,dXdt=\int_0^T\int_{\Omega_0}\frac{1}{2}|\nabla\phi|^2\,dxdt.$$

The least action principle (variations with respect to the domain deformations) yield

$$\tau = \nabla\phi \otimes \nabla\phi - \frac{1}{2}|\nabla\phi|^2,$$

and $\nabla \cdot \tau = \Delta\phi\nabla\phi$. This argument is also valid in the 3-dimensional case. Indeed,since $\nabla_i F_{ij} = 0$, we have that $F = \nabla \times A^j$ where A^j are divergent free vectors. Moreover, from the equation for F, these $A's$ will satisfy the transport equation of the form:

$$A_t - v \times (\nabla \times A) + \nabla b = 0,$$

where b is a scalar function. Hence in the cases of Ω being either the entire space or a periodic box,

$$\int_{\Omega_0} W(F)\,dX = \int_{\Omega_0} |F|^2\,dX = \int_{\Omega_0} \sum_{j=1}^{3} |\nabla \times A^j|^2\,dX. = \int_{\Omega_0} \sum_{j=1}^{3} |\nabla A^j|^2\,dX.$$

In the case that Ω is a bounded domain, since the boundary conditions of $v = 0$, and hence also $A = A_0$ are independent of the time t, we have

$$\begin{aligned} 2\int_{\Omega_0} W(F)\,dX &= \int_{\Omega} |F|^2\,dx = \int_{\Omega} |\nabla \times A|^2\,dx \\ &= \int_{\Omega} |\nabla A|^2 + \nabla \cdot ((\nabla A)A - (\nabla \cdot A)A)\,dx. \end{aligned}$$

We note that the last term on the right hand side of the above equation is a boundary integral which depends only on the Dirichlet condition of A [3, 6, 12]. Thus it is a fixed constant that determined by the initial condition A_0, and we have

$$2\int_{\Omega_0} W(F)\,dX = \int_{\Omega_0} |\nabla \times A|^2\,dX + C.$$

We then, as before, can derive the following induced elastic force

$$\nabla \cdot \tau = \Delta A \times (\nabla \times A) + \nabla\tilde{p},$$

through the variations of the flow maps $x(X,t)$, where $\tilde{p}$ is a scalar function.

From the energy law (2.9), we see that ϕ is a H^1 mapping from Ω into itself, with the Jacobian of the map never changing sign (being a constant). The following lemma gives the continuity of the mapping ϕ,though we shall not need it in this paper.

Lemma 2.1 *Let* $\mathbf{D}$ *be a two dimensional Lipschitz domain,* $\phi : \mathbf{D} \to \mathbf{D}$ *be a map satisfies:*

- $\phi(x) = x$*, on the boundary* $\partial\mathbf{D}$*;*
- $\det(\nabla\phi) = 1$*;*

- *$|\nabla\phi|_{L^2(\mathbf{D})} \leq M$ for some constant M.*

Then the following is true:

$$\sup_{x,y\in B_r(a)\cap\mathbf{D}} |\phi(x)-\phi(y)| \leq \frac{CM^{1/2}}{\sqrt{\log(1/r)}}. \tag{2.10}$$

Sketch of the proof. First, we assume that $B_{\sqrt{r}}(a) \subset \mathbf{D}$,$r$ is less than 1. Since $\phi \in H^1(\Omega)$,

$$\begin{aligned} M \geq \int_{B_{\sqrt{r}}(a)\setminus B_r(a)} |\nabla\phi|^2\,dx &= \int_{\sqrt{r}}^{r} \rho \int_{\partial B_\rho(a)} \frac{|\nabla\phi|^2}{\rho}\,dSd\rho \\ &\geq \frac{1}{2}\min_{\rho\in[r,\sqrt{r}]}(\rho\int_{\partial B_\rho(a)} |\nabla\phi|^2\,dS)\log\frac{1}{r}. \end{aligned}$$

In particular, there exists a $\rho_0 \in [r,\sqrt{r}]$, such that

$$\rho_0 \int_{\partial B_{\rho_0}(a)} |\nabla\phi|^2\,dS \leq \frac{4M}{\log(1/\rho_0)}.$$

On the other hand, from the Sobolev imbedding theorem on ∂B_{ρ_0}, we have

$$\sup_{x,y\in\partial B_{\rho_0}(a)\cap\mathbf{D}} |\phi(x)-\phi(y)|^2 \leq C\rho_0 \int_{\partial B_{\rho_0}(a)} |\nabla\phi|^2\,dS \leq C\frac{4M}{\log(1/\rho_0)}.$$

Since ϕ is incompressible, in particular, a local diffeomorphism on $\mathbf{D}$,

$$\sup_{x,y\in B_r(a)\cap\mathbf{D}} |\phi(x)-\phi(y)| \leq \sup_{x,y\in\partial B_{\rho_0}(a)} |\phi(x)-\phi(y)| \leq \frac{CM^{1/2}}{\sqrt{\log(1/r)}}.$$

If $a \in \partial\mathbf{D}$, then again we have a $\rho_0 \in [r,\sqrt{r}]$, such that

$$\rho_0 \int_{\partial(B_{\rho_0}(a)\cap\mathbf{D})} |\nabla\phi|^2\,dS \leq \frac{4M}{\log(1/\rho_0)}, \tag{2.11}$$

The similar argument as that for the interior case along with the continuity of the map at the boundary will lead to the desired result.

We note that it is classically well-known the above lemma is valid under much weak conditions. But we shall not elucidate it any further here. We also note that (2.6) is closely related to the Ericksen-Leslie systems, see for example [22]. Unlike [22], here we have, in addition, that $\nabla\cdot F$ vanishes.

2.2 Local existence of smooth solution

In this subsection, we are going to prove the local well-posedness of smooth enough solution to (2.6)–(2.8). The main result can be stated as follows:

Theorem 2.1 *Let $k \geq 2$ be a positive integer, $\nabla\phi_0 \in H^k(\Omega)$, $v_0 \in H^k(\Omega)$, then there exists a positive time T, which depends only on $|\nabla\phi_0|_{H^2}$ and $|v_0|_{H^2}$, such that the system (2.6)-(2.8) possesses a unique solution in the time interval $[0,T]$ with*

$$\begin{aligned} &\partial_t^j \nabla_x^\alpha v \in L^\infty([0,T];H^{k-2j-|\alpha|}(\Omega)) \cap L^2([0,T];H^{k-2j-|\alpha|+1}(\Omega)), \\ &\partial_t^j \nabla_x^\alpha \nabla\phi \in L^\infty([0,T];H^{k-2j-|\alpha|}(\Omega)), \end{aligned} \tag{2.12}$$

for all j, α satisfying $2j + |\alpha| \leq k$. Moreover, if T^* is the maximal time of existence, then

$$\int_0^{T^*} |\nabla v|_{H^2}^2 \, ds = +\infty. \tag{2.13}$$

Proof. We use Galerkin's approximation method to construct the approximate solutions to v equation, then use this approximate v and the transport equation to get the approximate solutions for ϕ. In the arguments for proving the convergence for the approximate solutions, it is essential to obtain a priori estimate for the approximate solutions. For simplicity, we will establish a priori estimates for the smooth solutions of (2.6)-(2.8). Therefore, let us assume in the rest of this subsection that (v,ϕ) is a local smooth solution to (2.6)-(2.8) on some time interval. As the proof is rather long, we divide it into two main steps.

step 1. H^2 estimate.

Integrate (2.9) over $[0,t]$, one obtains

$$\frac{1}{2}\int_\Omega \left(|v|^2 + |\nabla\phi|^2\right) dx + \mu \int_0^t \int_\Omega |\nabla v|^2 \, dx \, ds = \frac{1}{2}\int_\Omega \left(|v_0|^2 + |\nabla\phi_0|^2\right) dx. \tag{2.14}$$

Let us take the L^2 inner product of the second equation of (2.6) with v_t, observe that $v_t = 0$ on the boundary, we can use integration by parts to get

$$\begin{aligned} \frac{\mu}{2}\frac{d}{dt}|\nabla v|_{L^2}^2 + |v_t|^2 &= (v \otimes v, \nabla v_t) + (\nabla\phi \otimes \nabla\phi, \nabla v_t) \\ &\leq \left(|v|_{L^4}^2 + |\nabla\phi|_{L^4}^2\right) |\nabla v_t|_{L^2}. \end{aligned}$$

This together with the interpolations (in 2-D)

$$|v|_{L^4}^2 \leq C|v|_{L^2}|\nabla v|_{L^2}, \quad |\nabla\phi|_{L^4}^2 \leq C|\nabla\phi|_{L^2}^{\frac{3}{2}}|\nabla\triangle\phi|_{L^2}^{\frac{1}{2}}. \tag{2.15}$$

imply that

$$\frac{\mu}{2}\frac{d}{dt}|\nabla v|_{L^2}^2 + |v_t|^2 \leq C\left(|\nabla v|_{L^2}^2 + |\nabla\triangle\phi|_{L^2}\right) + \frac{\mu}{8}|\nabla v_t|_{L^2}^2. \tag{2.16}$$

While by taking t derivative of the second equation of (2.6), then taking the L^2 inner product of the resulting equation with v_t, and noticing again that $v_t = 0$ on the boundary, we can use integration by parts to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v_t|^2 + \mu|\nabla v_t|^2_{L^2} &= ((v_t \otimes v) + (\nabla\phi \otimes \nabla\phi)_t, \nabla v_t)\\ &\leq (|v|_{L^\infty}|v_t|_{L^2} + 2|\nabla\phi|_{L^\infty}|\nabla\phi_t|_{L^2})\,|\nabla v_t|_{L^2}.\end{aligned}$$

By the Sobolev imbedding

$$|v|_{L^\infty} \leq C(|v|_{L^2} + |\triangle v|_{L^2}), \quad |\nabla\phi|_{L^\infty} \leq C(\nabla\phi|_{L^2} + |\nabla\triangle\phi|_{L^2}), \tag{2.17}$$

we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v_t|^2 + \mu|\nabla v_t|^2_{L^2} \leq C((1 + |\triangle v|_{L^2})^2|v_t|^2_{L^2}&\\ +(1 + |\nabla\triangle\phi|_{L^2})^2|\nabla\phi_t|^2_{L^2}) + \frac{\mu}{4}|\nabla v_t|^2_{L^2},&\end{aligned} \tag{2.18}$$

where in the last step of the above derivation, we used the basic energy law (2.14). Next, by taking the L^2 inner product of the momentum equation with $\triangle v$ and using integration by parts again, we arrive at

$$\begin{aligned}\mu|\triangle v|^2_{L^2} &= (v_t, \triangle v) - (\nabla v \nabla v, \nabla v) + (\triangle\phi\nabla\phi, \triangle v)\\ &\leq C\left(|v_t|_{L^2}|\triangle v|_{L^2} + |v|^{\frac{1}{2}}_{L^2}|\nabla v|_{L^2}|\triangle v|^{\frac{3}{2}}_{L^2} + |\nabla\phi|_{L^\infty}|\nabla\phi|^{\frac{1}{2}}_{L^2}|\nabla\triangle\phi|^{\frac{1}{2}}_{L^2}|\triangle v|_{L^2}\right)\\ &\leq C\left(|v_t|^2_{L^2} + |\nabla v|^4_{L^2} + (1 + |\nabla\triangle\phi|_{L^2})^2|\nabla\triangle\phi|_{L^2}\right) + \frac{\mu}{2}|\triangle v|^2_{L^2},\end{aligned}$$

and hence

$$|\triangle v|^2_{L^2} \leq C\left(|v_t|^2_{L^2} + |\nabla v|^4_{L^2} + (1 + |\nabla\triangle\phi|_{L^2})^2|\nabla\triangle\phi|_{L^2}\right). \tag{2.19}$$

We now take one space derivative to the transport equation in (2.6),to deduce that

$$\begin{aligned}|\nabla\phi_t|_{L^2} &\leq C\left(|\nabla\phi|_{L^\infty}|\nabla v|_{L^2} + |v|_{L^\infty}|\nabla^2\phi|_{L^2}\right)\\ &\leq C\left((1 + |\nabla\triangle\phi|_{L^2})|\nabla v|_{L^2} + (1 + |\triangle v|_{L^2})|\nabla\triangle\phi|_{L^2}\right).\end{aligned} \tag{2.20}$$

Putting these estimates (2.19), (2.20) into the inequality (2.18), we obtain

$$\frac{d}{dt}|v_t|^2_{L^2} + \mu|\nabla v_t|^2_{L^2} \leq f_1(|\nabla v|_{L^2}, |v_t|_{L^2}, |\nabla\triangle\phi|_{L^2}), \tag{2.21}$$

where $f_1(\cdot,\cdot,\cdot)$ is a nonnegative and increasing function of its variables.

At this stage, it is clear we need to estimate third order derivatives of ϕ. We take triple derivatives to the transport equation of ϕ:

$$\nabla\Delta\phi_t + \nabla(v \cdot \nabla\Delta\phi) = -\nabla(\Delta v \cdot \nabla\phi + 2\nabla v \cdot \nabla\nabla\phi). \tag{2.22}$$

Multiply $\nabla\Delta\phi$ both sides, integrate over the domain, we get

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}|\nabla\Delta\phi|_{L^2}^2 = {} & -(\nabla\Delta v\nabla\phi + \nabla v\nabla\Delta\phi + \Delta v\nabla\nabla\phi + \\ & 2\nabla\nabla v\nabla\nabla\phi + 2\nabla v\nabla\nabla\nabla\phi, \nabla\Delta\phi)\end{aligned} \tag{2.23}$$

Now let us estimate the right hand side of the above equation term by term as follows:

$$|(\nabla\Delta v\nabla\phi, \nabla\Delta\phi)| \leq |\nabla\phi|_{L^\infty}|\nabla\Delta v|_{L^2}|\nabla\Delta\phi|_{L^2}, \tag{2.24}$$

$$|(\nabla v\nabla\Delta\phi, \nabla\Delta\phi)| \leq |\nabla v|_{L^\infty}|\nabla\Delta\phi|_{L^2}^2, \tag{2.25}$$

$$|(\Delta v\nabla\nabla\phi, \nabla\Delta\phi)| \leq |\nabla\Delta\phi|_{L^2}|\Delta v|_{L^4}|\nabla\nabla\phi|_{L^4}, \tag{2.26}$$

$$|(\nabla\nabla v\nabla\nabla\phi, \nabla\Delta\phi)| \leq |\nabla\Delta\phi|_{L^2}|\Delta v|_{L^4}|\nabla\nabla\phi|_{L^4}, \tag{2.27}$$

$$|(\nabla v\nabla\nabla\nabla\phi, \nabla\Delta\phi)| \leq |\nabla v|_{L^\infty}|\nabla\Delta\phi|_{L^2}^2. \tag{2.28}$$

Using again the interpolation $|\nabla\nabla\phi|_{L^4} \leq C|\nabla\phi|_{L^2}^{1/4}|\nabla\Delta\phi|_{L^2}^{3/4}$, and combining these inequalities, we have

$$\frac{d}{dt}|\nabla\Delta\phi|_{L^2}^2 \leq C\left((1 + |\nabla\Delta v|_{L^2})|\nabla\Delta\phi|_{L^2}^2 + |\nabla\Delta v|_{L^2}^{\frac{3}{4}}|\nabla\Delta\phi|_{L^2}^{\frac{7}{4}}\right). \tag{2.29}$$

Next, by applying ∇_x to the momentum equation of (2.6), and taking the L^2 inner product of the resulting equation with $\nabla\Delta v$, we obtain

$$\begin{aligned}\mu|\nabla\Delta v|_{L^2}^2 &= (\nabla v_t, \nabla\Delta v) + (\nabla(v\nabla v), \nabla\Delta v) + (\nabla(\Delta\phi\nabla\phi), \nabla\Delta v) \\ &\leq \left(|\nabla v_t|_{L^2} + |v|_{L^\infty}|\Delta v|_{L^2} + |\nabla v|_{L^4}^2 + |\nabla\phi|_{L^\infty}|\nabla\Delta\phi|_{L^2} + |\Delta\phi|_{L^4}^2\right)|\nabla\Delta v|_{L^2} \\ &\leq C\left(|\nabla v_t|_{L^2} + |v|_{H^2}|\Delta v|_{L^2} + (|\nabla\phi|_{L^2} + |\nabla\Delta\phi|_{L^2})|\nabla\Delta\phi|_{L^2}\right)|\nabla\Delta v|_{L^2},\end{aligned}$$

which implies that

$$\begin{aligned}|\nabla\Delta v|_{L^2}^2 \leq C\{&|\nabla v_t|^2 + (1 + |\Delta v|_{L^2})^2|\Delta v|_{L^2}^2 \\ &+(1 + |\nabla\Delta\phi|_{L^2})^2|\nabla\Delta\phi|_{L^2}\}.\end{aligned} \tag{2.30}$$

Substituting (2.19) and (2.30) into (2.29), we find

$$\frac{d}{dt}|\nabla\Delta\phi|_{L^2}^2 \leq f_2(|\nabla v|_{L^2}, |v_t|_{L^2}, |\nabla\Delta\phi|_{L^2}) + \frac{\mu}{8}|\nabla v_t|_{L^2}^2, \tag{2.31}$$

where f_2 has exactly the same properties as f_1 in (2.21).

Combining (2.16), (2.21) and (2.31), we finally arrive at

$$\begin{aligned}\frac{d}{dt}\left(\mu|\nabla v|_{L^2}^2 + |v_t|_{L^2}^2 + |\nabla\Delta\phi|_{L^2}^2\right) + |v_t|_{L^2}^2 + \mu|\nabla v_t|_{L^2}^2 & \\ \leq f(|\nabla v|_{L^2}, |v_t|_{L^2}, |\nabla\Delta\phi|_{L^2}),&\end{aligned} \tag{2.32}$$

Since $\nabla \cdot v_t = 0$, we see, by the second equation of (2.6), that

$$|v_t(x,0)|_{L^2} \le C(|v_0|_{H^2}, |\nabla\phi_0|_{H^2}). \tag{2.33}$$

This last estimate together with (2.32) and the Gronwall inequality yield that there exist positive constants T, M_2 which depend only on $|v_0|_{H^2} + |\nabla\phi_0|_{H^2}$, such that

$$|\nabla v|_{L^2}^2 + |v_t|_{L^2}^2 + |\nabla\Delta\phi|_{L^2}^2 + \int_0^t \left(|v_t|_{L^2}^2 + \mu|\nabla v_t|_{L^2}^2\right) ds \le M_2, \tag{2.34}$$

for $0 \le t \le T$.

Using the above estimate, we may deduce from (2.19), (2.20) and (2.30), that

$$|\Delta v|_{L^2}^2 + |\nabla\phi_t|_{L^2}^2 + \int_0^t |\nabla\Delta v|_{L^2}^2 \, ds \le M_2, \quad 0 \le t \le T. \tag{2.35}$$

This proves (2.12) for $k = 2$,and that is the most important item in our proof of the existence theorem.

step 2. Higher order energy estimate.

In this step, we are going to prove inductively that

$$|\partial_t^j \nabla_x^\alpha v|_{L^2} + |\partial_t^j \nabla_x^\alpha \nabla\phi|_{L^2} + \int_0^t |\partial_t^j \nabla_x^\alpha \nabla v|_{L^2} \, ds \le M_k(|v_0|_{H^k}, |\nabla\phi_0|_{H^k}), \tag{2.36}$$

for all j, α satisfying $2j + |\alpha| \le k$ and $0 \le t \le T$, with T being determined in (2.34). With the result in step 1, we assume (2.36) holds for all j and α satisfying $2 \le 2j + |\alpha| \le k-1$, we are going to prove that (2.36) holds for all j and α satisfying $2 \le 2j + |\alpha| \le k$. If k is an odd number, we denote $j_1 = \frac{k-1}{2}, j_3 = \frac{k-3}{2}, \cdots$, if k is an even number, we denote $j_0 = \frac{k}{2}, j_2 = \frac{k-2}{2}, \cdots$. Before proceeding any further, we apply ∂_t^j to (2.6) to get

$$\partial_t^{j+1}\nabla\phi + \partial_t^j(v \cdot \nabla\nabla\phi) + \partial_t^j(\nabla v \cdot \nabla\phi) = 0, \tag{2.37}$$
$$\partial_t^{j+1} v + \partial_t^j(v\nabla v) + \nabla\partial_t^j p = \mu\Delta\partial_t^j v - \partial_t^j(\Delta\phi\nabla\phi). \tag{2.38}$$

step 2.1. If k is an odd number, let us take $j = j_1$ in (2.38), and taking the L^2 inner product of the resulting equation with $\partial_t^{j_1+1} v$, then since $\partial_t^{j_1} v = 0$ on the boundary, we can use integration by parts to get

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}|\nabla\partial_t^{j_1} v|_{L^2}^2 + |\partial_t^{j_1+1} v|_{L^2}^2 &= -(\partial_t^{j_1}(v\nabla v), \partial_t^{j_1+1} v) - (\partial_t^{j_1}(\Delta\phi\nabla\phi), \partial_t^{j_1+1} v)\\ &\le 4\left(|\partial_t^{j_1}(v\nabla v)|_{L^2}^2 + |\partial_t^{j_1}(\Delta\phi\nabla\phi)|_{L^2}^2\right) + \frac{1}{2}|\partial_t^{j_1+1} v|_{L^2}^2.\end{aligned} \tag{2.39}$$

Note also

$$|\partial_t^{j_1}(v\nabla v)|_{L^2} \le \sum_{m_1+m_2=j_1} |\partial_t^{m_1} v \partial_t^{m_2} \nabla v|_{L^2}. \tag{2.40}$$

If m_1 or $m_2 = 0$,

$$|v\partial_t^{j_1}\nabla v|_{L^2} \leq |v|_{L^\infty}|\partial_t^{j_1}\nabla v|_{L^2}, \quad |\partial_t^{j_1}v\nabla v|_{L^2} \leq |\nabla v|_{L^\infty}|\partial_t^{j_1}v|_{L^2}. \tag{2.41}$$

Since $2(j_1-1)+2 \leq k-1$ by the definition, we deduce, by the inductive assumption, that

$$|\partial_t^{m_1}v\partial_t^{m_2}\nabla v|_{L^2} \leq C|\partial_t^{m_1}v|_{L^2}^{\frac{1}{2}}|\partial_t^{m_1}\nabla v|_{L^2}^{\frac{1}{2}}|\partial_t^{m_2}v|_{L^2}^{\frac{1}{2}}|\partial_t^{m_2}\nabla v|_{L^2}^{\frac{1}{2}} \leq M_{k-1}. \tag{2.42}$$

Summing up the above, we obtain

$$|\partial_t^{j_1}(v\nabla v)|_{L^2} \leq M_{k_1}\left(1+|\partial_t^{j_1}\nabla v|_{L^2}+|\nabla\Delta v|_{L^2}\right). \tag{2.43}$$

By a similar argument, we also have

$$\begin{aligned}
|\partial_t^{j_1}(\Delta\phi\nabla\phi)|_{L^2} \leq\ & |\nabla\phi|_{L^\infty}|\partial_t^{j_1}\Delta\phi|_{L^2} \\
& + \sum_{m_1+m_2=j_1, 1\leq m_1\leq j_1-1} |\partial_t^{m_1}\nabla\phi|_{L^4}|\partial_t^{m_2}\Delta\phi|_{L^4} + |\partial_t^{j_1}\nabla\phi|_{L^4}|\Delta\phi|_{L^4} \\
\leq\ & M_{k-1}\left(1+|\partial_t^{j_1}\Delta\phi|_{L^2}+|\partial_t^{j_1}\Delta\phi|_{L^2}^{\frac{1}{2}}|\partial_t^{j_1}\nabla\phi|_{L^2}^{\frac{1}{2}}\right) \\
\leq\ & M_{k-1}\left(1+|\partial_t^{j_1}\Delta\phi|_{L^2}\right).
\end{aligned} \tag{2.44}$$

Combining (2.39) through (2.44), we conclude that

$$\mu\frac{d}{dt}|\nabla\partial_t^{j_1}v|_{L^2}^2 + |\partial_t^{j_1+1}v|_{L^2}^2 \leq M_{k-1}\left(1+|\partial_t^{j_1}\nabla v|_{L^2}+|\partial_t^{j_1}\Delta\phi|_{L^2}+|\nabla\Delta v|_{L^2}\right)^2. \tag{2.45}$$

On the other hand, by taking $j = j_1$ in (2.37), then applying $\nabla\cdot$ to the resulting equation, we find

$$\frac{1}{2}\frac{d}{dt}|\partial_t^{j_1}\Delta\phi|_{L^2}^2 + \left(\partial_t^{j_1}(v\cdot\nabla\Delta\phi)+\partial_t^{j_1}(\nabla v\nabla^2\phi)+\partial_t^{j_1}\nabla(\nabla v\nabla\phi), \partial_t^{j_1}\Delta\phi\right) = 0. \tag{2.46}$$

In particular for $j_1 = 1$ in (2.46), we get

$$\begin{aligned}
|(\partial_t(v\nabla\triangle\phi), \partial_t\Delta\phi)| &= |(\partial_t v\nabla\triangle\phi, \partial_t\Delta\phi)| \\
&\leq |\partial_t v|_{L^\infty}|\nabla\Delta\phi|_{L^2}|\partial_t\Delta\phi|_{L^2} \\
&\leq M_2(|\partial_t v|_{L^2}+|\partial_t\Delta v|_{L^2})|\partial_t\Delta\phi|_{L^2}.
\end{aligned} \tag{2.47}$$

While by taking $j_1 = 1$ in (2.38), and taking the L^2 inner product of the resulting equation with Δv_t, we obtain

$$\mu|\Delta v_t|_{L^2}^2 = (v_{tt}, \Delta v_t) + ((v\nabla v)_t, \Delta v_t) + ((\Delta\phi\nabla\phi)_t, \Delta v_t). \tag{2.48}$$

This leads us to

$$\begin{aligned}
|\Delta v_t|_{L^2}^2 &\leq C\left(|v_{tt}|_{L^2}^2+|(v\nabla v)_t|_{L^2}^2+|(\Delta\phi\nabla\phi)_t|_{L^2}^2\right) \\
&\leq M_2\left(|v_{tt}|_{L^2}^2+|\nabla\Delta v|_{L^2}^2+|\nabla v_t|_{L^2}+|\Delta\phi_t|_{L^2}\right).
\end{aligned} \tag{2.49}$$

Substituting (2.49) into (2.47), we obtain

$$|(\partial_t(v\nabla\triangle\phi),\partial_t\Delta\phi)| \le M_2\left(1+|v_{tt}|_{L^2}+|\nabla\Delta v|_{L^2}+|\nabla v_t|_{L^2}+|\Delta\phi_t|_{L^2}\right)|\Delta\phi_t|_{L^2}. \quad (2.50)$$

Using (2.49) again, we finally deduce

$$\begin{aligned}|(\partial_t(\nabla v\nabla^2\phi),\partial_t\Delta\phi)| &\le |\nabla v|_{L^\infty}|\Delta\phi_t|_{L^2}^2+|\nabla v_t|_{L^4}|\nabla^2\phi|_{L^4}|\Delta\phi_t|_{L^2}\\ &\le |\nabla v|_{L^\infty}|\Delta\phi_t|_{L^2}^2+M_2\left(|\nabla v_t|_{L^2}^2+|\Delta\phi_t|_{L^2}^2+|\nabla\Delta v|_{L^2}^2\right)\\ &\quad +\frac{1}{4}|v_{tt}|_{L^2}^2, \end{aligned} \quad (2.51)$$

Following an almost identical argument as that we used in deriving the estimates (2.50) and (2.51), we can deduce a desired estimate for the last term in (2.46). Summing up (2.46), (2.50) and (2.51) together, we obtain

$$\frac{d}{dt}|\Delta\phi_t|_{L^2}^2 \le M_2\left(1+|\nabla\Delta v|_{L^2}^2+|\nabla v_t|_{L^2}^2+(1+|\nabla v|_{L^\infty})|\Delta\phi_t|_{L^2}^2\right)+\frac{1}{2}|v_{tt}|_{L^2}^2. \quad (2.52)$$

Take $j_1=1$ in (2.45), and combine it with (2.52), we have

$$\begin{aligned}&\frac{d}{dt}\left(\mu|\nabla v_t|_{L^2}^2+|\Delta\phi_t|_{L^2}^2\right)+\frac{1}{2}|v_{tt}|_{L^2}^2\\ &\le M_2\left(1+|\nabla\Delta v|_{L^2}^2+|\nabla v_t|_{L^2}^2+(1+|\nabla v|_{L^\infty})|\Delta\phi_t|_{L^2}^2\right). \end{aligned} \quad (2.53)$$

We can also deduce from (2.35) that $\int_0^T|\nabla v|_{L^\infty}\,ds<M_2$. Since

$$|\Delta\phi_t(x,0)|_{L^2}=|\Delta(v_0\nabla\phi_0)|_{L^2}\le|v_0|_{H^2}|\nabla\phi_0|_{H^2}, \quad (2.54)$$

one can apply the Gronwall inequality to obtain that

$$|\nabla v_t|_{L^2}^2+|\Delta\phi_t|_{L^2}^2+\int_0^t|v_{tt}|_{L^2}^2\le M_3,\quad 0\le t\le T. \quad (2.55)$$

This along with (2.30) and (2.49) yield that

$$|\nabla\Delta v|_{L^2}^2+\int_0^t|\Delta v_t|_{L^2}^2\,ds\le M_3,\qquad 0\le t\le T. \quad (2.56)$$

As for the proof of (2.30), one can use (2.56) to conclude

$$\begin{aligned}|\Delta^2 v|_{L^2}&\le C\left(|\Delta v_t|_{L^2}+|v|_{L^\infty}|v|_{H^3}+|\nabla\phi|_{L^\infty}|\nabla\phi|_{H^3}\right)\\ &\le C\left(|\Delta v_t|_{L^2}+M_3(1+|\nabla\phi|_{H^3})\right). \end{aligned} \quad (2.57)$$

Similarly, by the transport equation for ϕ, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla^4\phi|_{L^2}^2&=-\left(\nabla^3(v\nabla\nabla\phi)-v\nabla\nabla^4\phi,\nabla^4\phi\right)-\left(\nabla^3(\nabla v\nabla\phi),\nabla^4\phi\right)\\ &\le M_3\left(1+|\nabla^4\phi|_{L^2}^2+|\Delta^2 v|_{L^2}^2\right). \end{aligned} \quad (2.58)$$

(2.55), (2.57) together with Gronwall inequality imply that

$$|\nabla^4\phi|_{L^2}^2 + \int_0^t |\Delta^2 v|_{L^2}^2 \, ds \le M_3, \quad 0 \le t \le T. \tag{2.59}$$

By (2.55), (2.56) and (2.59), we have thus proved (2.36) for $k = 3$. Following the same procedure, we can prove (2.36) for k being general odd number.

step 2.2. If k is an even number, let us set $j = j_0$ in (2.38), and take the L^2 product of the resulting equation with $\partial_t^{j_0} v$, again by the boundary condition that $\partial_t^{j_0} v = 0$ on the boundary, we use integration by parts to get

$$\frac{1}{2}\frac{d}{dt}|\partial_t^{j_0} v|_{L^2}^2 + \mu|\nabla\partial_t^{j_0} v|_{L^2}^2 = \left(\partial_t^{j_0}(v\otimes v + \nabla\phi\otimes\nabla\phi), \nabla\partial_t^{j_0} v\right), \tag{2.60}$$

and thus

$$\frac{d}{dt}|\partial_t^{j_0} v|_{L^2}^2 + \mu|\nabla\partial_t^{j_0} v|_{L^2}^2 \le \frac{1}{\mu}\left(|\partial_t^{j_0}(v\otimes v)|_{L^2}^2 + |\partial_t^{j_0}(\nabla\phi\otimes\nabla\phi)|_{L^2}^2\right). \tag{2.61}$$

Next let us estimate the right hand side term by term. First of all,

$$|\partial_t^{j_0}(v\otimes v)|_{L^2} \le \sum_{m_1+m_2=j_0} |\partial_t^{m_1} v \partial^{m_2} v|_{L^2}. \tag{2.62}$$

From (2.35), one has that $|v|_{L^\infty} \le M_2$ for $0 \le t \le T$. If m_1 or m_2 equals 0, then

$$|v\partial_t^{j_0} v|_{L^2} \le M_2 |\partial_t^{j_0} v|_{L^2}, \tag{2.63}$$

otherwise,

$$\begin{aligned} |\partial_t^{m_1} v \partial^{m_2} v|_{L^2} &\le |\partial_t^{m_1} v|_{L^4} |\partial^{m_2} v|_{L^4} \\ &\le C |\partial_t^{m_1} v|_{L^2}^{\frac{1}{2}} |\nabla\partial_t^{m_1} v|_{L^2}^{\frac{1}{2}} |\partial_t^{m_2} v|_{L^2}^{\frac{1}{2}} |\nabla\partial_t^{m_2} v|_{L^2}^{\frac{1}{2}}. \end{aligned} \tag{2.64}$$

Note that $1 \le m_i \le j_0 - 1, 2m_i + 1 \le k_1$, therefore by the inductive assumption, we arrive at

$$|\partial_t^{m_1} v \partial^{m_2} v|_{L^2}^2 \le M_{k-1}(1 + |\nabla\partial_t^{j_0-1} v|_{L^2}^2). \tag{2.65}$$

To summarize, we have so far obtained

$$|\partial_t^{m_1} v \partial^{m_2} v|_{L^2}^2 \le M_2\left(1 + |\partial_t^{j_0} v|_{L^2}^2 + |\nabla\partial_t^{j_0-1} v|_{L^2}^2\right) \tag{2.66}$$

Next we note, as for (2.66), that $|\nabla\phi|_{L^\infty} \le M_2$, $j_0 \le k-1$ for $k \ge 2$, we find

$$|\partial_t^{j_0}(\nabla\phi\otimes\nabla\phi)|_{L^2} \le M_{k-1}\left(1 + |\partial_t^{j_0}\nabla\phi|_{L^2}\right). \tag{2.67}$$

Combining (2.61), (2.66) and (2.67), we find

$$\frac{d}{dt}|\partial_t^{j_0} v|_{L^2}^2 + \mu|\nabla\partial_t^{j_0} v|_{L^2}^2 \le M_{k-1}\left(1 + |\partial_t^{j_0} v|_{L^2}^2 + |\nabla\partial_t^{j_0-1} v|_{L^2}^2 + |\partial_t^{j_0}\nabla\phi|_{L^2}\right). \tag{2.68}$$

On the other hand, by taking $j = j_0$ in (2.37), and taking the L^2 inner product of the resulting equation with $\partial_t^{j_0}\nabla\phi$, we get

$$\frac{1}{2}\frac{d}{dt}|\partial_t^{j_0}\nabla\phi|_{L^2}^2 = -(\partial_t^{j_0}(v\nabla\nabla\phi), \partial_t^{j_0}\nabla\phi) - (\partial_t^{j_0}(\nabla v\nabla\phi), \partial_t^{j_0}\nabla\phi). \tag{2.69}$$

Again for a clear presentation of the main idea, we take $j_0 = 2$ in the above, then by integration by parts, we get

$$\begin{aligned} |(\partial_t^2(v\nabla\nabla\phi), \partial_t^2\nabla\phi)| &= |(\partial_t^2(v\nabla\nabla\phi) - v\nabla\partial_t^2\nabla\phi, \partial_t^2\nabla\phi)| \\ &\le C\left(|\nabla^2\phi|_{L^\infty}|v_{tt}|_{L^2} + |v_t|_{L^\infty}|\partial_t^2\nabla\phi|_{L^2}\right)|\partial_t^2\nabla\phi|_{L^2}. \end{aligned}$$

This together with (2.55) and (2.59) imply that

$$|(\partial_t^2(v\nabla\nabla\phi), \partial_t^2\nabla\phi)| \le M_3\left(|\partial_t^2 v|_{L^2} + |v_t|_{L^\infty}\right)|\partial_t^2\nabla\phi|_{L^2}. \tag{2.70}$$

The second term on the right hand side of (2.69) can be estimated as follows:

$$\begin{aligned} |(\partial_t^{j_0}(\nabla v\nabla\phi), \partial_t^{j_0}\nabla\phi)| \le C\{&|\nabla\phi|_{L^\infty}|\partial_t^2\nabla v|_{L^2} \\ &+|\nabla v_t|_{L^2}^{\frac{1}{2}}|\Delta v_t|_{L^2}^{\frac{1}{2}}|\nabla\phi_t|_{L^2}^{\frac{1}{2}}|\Delta\phi_t|_{L^2}^{\frac{1}{2}} \\ &+|\nabla v|_{L^\infty}|\partial_t^2\nabla\phi|_{L^2}\}|\partial_t^2\nabla\phi|_{L^2}. \end{aligned} \tag{2.71}$$

This last inequality together with (2.20) and (2.55) lead to

$$|(\partial_t^{j_0}(\nabla v\nabla\phi), \partial_t^{j_0}\nabla\phi)| \le M_3\left(1 + |\partial_t^2\nabla\phi|_{L^2} + |\Delta v_t|_{L^2}\right) + \frac{\mu}{4}|\partial_t^2\nabla v|_{L^2}^2. \tag{2.72}$$

By summarizing estimates (2.69) through (2.72), we obtain

$$\frac{d}{dt}|\partial_t^2\nabla\phi|_{L^2}^2 \le M_3\left(1 + |v_{tt}|_{L^2}^2(1 + |v_t|_{L^\infty})|\partial_t^2\nabla\phi|_{L^2}^2 + |\Delta v_t|_{L^2}\right) + \frac{\mu}{4}|\partial_t^2\nabla v|_{L^2}^2. \tag{2.73}$$

Finally by taking $j_0 = 2$ in (2.68), and using (2.73), we get

$$\begin{aligned} &\frac{d}{dt}\left(|\partial_t^2 v|_{L^2}^2 + |\partial_t^2\nabla\phi|_{L^2}^2\right) + \mu|\nabla\partial_t^2 v|_{L^2}^2 \\ &\le M_3\left(1 + |v_{tt}|_{L^2}^2(1 + |v_t|_{L^\infty})|\partial_t^2\nabla\phi|_{L^2}^2 + |\Delta v_t|_{L^2}\right). \end{aligned} \tag{2.74}$$

Gronwall inequality together with (2.56) and (2.74) yield

$$|\partial_t^2 v|_{L^2}^2 + |\partial_t^2\nabla\phi|_{L^2}^2 + \int_0^t |\nabla\partial_t^2 v|_{L^2}^2\, ds \le M_4, \quad 0 \le t \le T. \tag{2.75}$$

Following the exactly same procedure as that from (2.56) to (2.59), we can prove

$$|\Delta v_t|_{L^2}^2 + |\Delta^2 v|_{L^2}^2 + |\Delta\nabla\phi_t|_{L^2} + |\nabla^5\phi|_{L^2}^2 \le M_4, \quad 0 \le t \le T. \tag{2.76}$$

This proves (2.36) for $k = 4$. Similar to the above procedure, we can prove (2.36)for k being a general even number.

The proof itself implies that there holds (2.13), therefore we have completed the proof of the Theorem.

Remark. The above results also holds for the 3-dimensional case. The differences in the estimates are that, instead of (2.15), we will need following interpolation inequalities:

$$|v|_{L^4}^2 \leq C|v|_{L^2}^{1/2}|\nabla v|_{L^2}^{3/2}, \quad |\nabla A|_{L^4}^2 \leq C|\nabla A|_{L^2}^{\frac{5}{4}}|\nabla \triangle A|_{L^2}^{\frac{3}{4}},$$

and also the inequality :

$$|\nabla\nabla A|_{L^4} \leq C|\nabla A|_{L^2}^{\frac{1}{8}}|\nabla \triangle A|_{L^2}^{\frac{7}{8}}.$$

We should also point out that the above local time existence is still valid for the Oldroyd systems in variables (v, F) by the same type of estimates. In fact, our proof works even for the cases of Oldroyd systems with general initial data (without the divergent zero condition on F).

2.3 Global existence for small data

The main difficulty in our system is the lackness of the damping mechanism on F (or in ϕ in our formulation). This is different from the cases studied in [20] where the contribution of the strain rate (symmetric part of ∇v) in the constitutive equation is ignored, [2] and many others where a linear damping term is present.

Realizing the above difficulty and at the same time, noticing that the induced stress term $\Delta\phi\nabla\phi$ in the momentum equation is derived from the energy, we believe it should provide a certain dissipation to the system. Here we will restrict ourself to the case of small datum, namely, ϕ is closed to the identity map (F is closed to the identity matrix). Our goal is to prove the global existence of solutions with small initial data. To make a clear presentation of the main idea, we will only consider the case that Ω is either a periodic box or the whole space R^2. The Dirichlet boundary case is studied in our forthcoming work,see also remarks at the end of this subsection.

We introduce the perturbation of the identity map

$$\psi(x) = \phi(x) - x. \tag{2.77}$$

Since ϕ is a volume preserving map from Ω to itself, i.e. $\det(\nabla\phi) = 1$, ψ will be a map with $\operatorname{div}\psi = \operatorname{tr}(\nabla\psi)$ being relatively small (in the order of $|\nabla\psi|^2$).

We can rewrite the system (2.6) into:

$$\begin{cases} \psi_t + v \cdot \nabla\psi = -v, \\ v_t + v \cdot \nabla v + \nabla p = \mu\Delta v - \Delta\psi - \Delta\psi\nabla\psi, \\ \nabla \cdot v = 0. \end{cases} \tag{2.78}$$

The initial conditions become:

$$\psi(x, 0) = \psi_0, \; v(x, 0) = v_0(x), \tag{2.79}$$

with condition $(\nabla\psi_0 + I) = 1$.

Set $w = v - \frac{1}{\mu}\psi$, and by subtracting the two equations,we have

$$w_t - \mu\Delta w = -v\cdot\nabla w - \nabla p - \Delta\psi\nabla\psi + \frac{v}{\mu}. \tag{2.80}$$

The basic energy conservation law reads:

$$\frac{1}{2}\frac{d}{dt}(|v|_{L^2}^2 + |\nabla\psi|_{L^2}^2) + \mu|\nabla v|_{L^2}^2 = 0. \tag{2.81}$$

Next we apply $\nabla\Delta$ to the both sides of the ψ equation in system (2.78), and multiply the resulting equation by $\nabla\Delta\psi$, integrate over Ω to get:

$$\frac{1}{2}\frac{d}{dt}|\nabla\Delta\psi|_{L^2}^2 + (\nabla\Delta(v\cdot\nabla\psi), \nabla\Delta\psi) = -(\nabla\Delta v, \nabla\Delta\psi). \tag{2.82}$$

The Moser type inequality (see P. 43 of [23]) implies,

$$\begin{aligned}|(\nabla\Delta(v\cdot\nabla\psi), \nabla\Delta\psi)| &= |(\nabla\Delta(v\cdot\nabla\psi) - v\cdot\nabla\nabla\Delta\psi, \nabla\Delta\psi)|\\ &\le C(|\nabla v|_{L^\infty}|\nabla\Delta\psi|_{L^2}^2 + |\nabla\psi|_{L^\infty}|\nabla\Delta v|_{L^2}|\nabla\Delta\psi|_{L^2}),\end{aligned}$$

then by (2.17) and the fact that $\psi = \mu(v - w)$, we obtain

$$\begin{aligned}|(\nabla\Delta(v\cdot\nabla\psi), \nabla\Delta\psi)| &\le C((|\nabla v|_{L^2} + |\nabla\Delta v|_{L^2})|\nabla\Delta\psi|_{L^2}^2 + |\nabla\psi|_{H^2}|\nabla\Delta v|_{L^2}|\nabla\Delta\psi|_{L^2})\\ &\le C\mu|\nabla\psi|_{H^2}(|\nabla v|_{L^2}^2 + |\nabla\Delta v|_{L^2}^2 + |\nabla\Delta w|_{L^2}^2). \qquad (2.83)\end{aligned}$$

This leads to

$$\frac{1}{2}\frac{d}{dt}|\nabla\Delta\psi|_{L^2}^2 + (\nabla\Delta v, \nabla\Delta\psi) \le C\mu|\nabla\psi|_{H^2}(|\nabla v|_{L^2}^2 + |\nabla\Delta v|_{L^2}^2 + |\nabla\Delta w|_{L^2}^2). \tag{2.84}$$

Similarly, after applying Δ to the v equation in the system (2.78), multiplying by Δv and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}|\Delta v|_{L^2}^2 + \mu|\nabla\Delta v|_{L^2}^2 = -(\Delta(v\cdot\nabla v), \Delta v) - (\Delta^2\psi, \Delta v) - (\Delta(\Delta\psi\nabla\psi), \Delta v). \tag{2.85}$$

Now let us estimate the right hand side term by term. First of all,by (2.15), we get

$$\begin{aligned}|(\Delta(v\cdot\nabla v), \Delta v)| &= |(\nabla(v\cdot\nabla v), \nabla\Delta v)|\\ &\le |\nabla v|_{L^4}^2|\nabla\Delta v|_{L^2} + |v|_{L^4}|\Delta v|_{L^4}|\nabla\Delta v|_{L^2}\\ &\le C\left(|\nabla v|_{L^2}|\nabla\Delta v|_{L^2}^2 + |v|_{L^2}^{1/2}|\nabla v|_{L^2}^{1/2}|\Delta v|_{L^2}^{1/2}|\nabla\Delta v|_{L^2}^{1/2}|\nabla\Delta v|_{L^2}\right)\\ &\le C\{|\nabla v|_{L^2}|\nabla\Delta v|_{L^2}^2 + |v|_{L^2}|\nabla v|_{L^2}^2\\ &\quad + |v|_{L^2}|\nabla\Delta v|_{L^2}^2 + |\Delta v|_{L^2}|\nabla\Delta v|_{L^2}^2\}\\ &\le C(|v|_{L^2}|\nabla v|_{L^2}^2 + |v|_{H^2}|\nabla\Delta v|_{L^2}^2). \qquad (2.86)\end{aligned}$$

Next, integration by parts, one has

$$(\Delta^2\psi, \Delta v) = -(\nabla\Delta\psi, \nabla\Delta v). \tag{2.87}$$

Finally, using the fact that $\psi = \mu(v-w)$, we have

$$\begin{aligned}|(\Delta(\Delta\psi\nabla\psi), \Delta v)| &= |(\nabla(\Delta\psi\nabla\psi), \nabla\Delta v)| \\ &\le |\nabla\psi|_{L^\infty}|\nabla\Delta\psi|_{L^2}|\nabla\Delta v|_{L^2} + |\Delta\psi|_{L^4}^2|\nabla\Delta v|_{L^2} \\ &\le (|\nabla\psi|_{L^\infty} + |\Delta\psi|_{L^2})|\nabla\Delta\psi|_{L^2}|\nabla\Delta v|_{L^2} \\ &\le C\mu|\nabla\psi|_{H^2}(|\nabla\Delta v|_{L^2}^2 + |\nabla\Delta w|_{L^2}^2).\end{aligned} \tag{2.88}$$

Substitute these results into (2.85), one concludes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\Delta v|_{L^2}^2 + \mu|\nabla\Delta v|_{L^2}^2 &\le (\nabla\Delta\psi, \nabla\Delta v) \\ &+ C(|v|_{L^2}|\nabla v|_{L^2}^2 + (|v|_{H^2} + \mu|\nabla\psi|_{H^2})|\nabla\Delta v|_{L^2}^2 + \mu|\nabla\psi|_{H^2}|\nabla\Delta w|_{L^2}^2).\end{aligned} \tag{2.89}$$

While by applying the Laplacian to the both sides of (2.80), then multiplying by Δw, and integrating over the domain, we have

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}|\Delta w|_{L^2}^2 + \mu|\nabla\Delta w|_{L^2}^2 \\ &= (\Delta(-v\cdot\nabla w - \nabla p - \Delta\psi\nabla\psi + \frac{v}{\mu}), \Delta w).\end{aligned} \tag{2.90}$$

To estimate the right hand side of the above equation, we note, by (2.15), that

$$\begin{aligned}|(\Delta(v\cdot\nabla w), \Delta w)| &= |(\nabla(v\cdot\nabla w), \nabla\Delta w)| \\ &\le |\nabla v|_{L^\infty}|\nabla w|_{L^2}|\nabla\Delta w|_{L^2} + |v|_{L^4}|\Delta w|_{L^4}|\nabla\Delta w|_{L^2} \\ &\le C\,(|\nabla v|_{L^2} + |\nabla\Delta v|_{L^2})|\nabla w|_{L^2}|\nabla\Delta w|_{L^2} \\ &\quad + |v|_{L^2}^{1/2}|\nabla v|_{L^2}^{1/2}|\Delta w|_{L^2}^{1/2}|\nabla\Delta w|_{L^2}^{1/2}|\nabla\Delta w|_{L^2}\Big) \\ &\le C\,\big((|v|_{L^2} + |\nabla w|_{L^2})|\nabla v|_{L^2}^2 \\ &\quad + (|v|_{L^2} + |\nabla w|_{L^2} + |\Delta w|_{L^2})|\nabla\Delta w|_{L^2}^2 \\ &\quad + |\nabla w|_{L^2}|\nabla\Delta v|_{L^2}^2\big)\,.\end{aligned} \tag{2.91}$$

Similarly one has

$$\begin{aligned}|(\Delta(\Delta\psi\nabla\psi), \Delta w)| &= |(\nabla(\Delta\psi\nabla\psi), \nabla\Delta w)| \\ &\le |\nabla\psi|_{L^\infty}|\nabla\Delta\psi|_{L^2}|\nabla\Delta w|_{L^2} + |\Delta\psi|_{L^4}^2|\nabla\Delta w|_{L^2} \\ &\le (|\nabla\psi|_{L^\infty} + |\Delta\psi|_{L^2})|\nabla\Delta\psi|_{L^2}|\nabla\Delta w|_{L^2} \\ &\le C\mu(|\nabla\psi|_{L^\infty} + |\Delta\psi|_{L^2})(|\nabla\Delta v|_{L^2}|\nabla\Delta w|_{L^2} + |\nabla\Delta w|_{L^2}^2) \\ &\le C\mu(|\nabla\psi|_{H^2}|\nabla\Delta v|_{L^2}^2 + |\nabla\psi|_{H^2}|\nabla\Delta w|_{L^2}^2).\end{aligned} \tag{2.92}$$

and

$$|(\frac{\Delta v}{\mu},\Delta w)|=\frac{1}{\mu}|(\nabla v,\nabla\Delta w)|\le\frac{1}{2\mu^3}|\nabla v|_{L^2}^2+\frac{\mu}{2}|\nabla\Delta w|_{L^2}^2. \tag{2.93}$$

On the other hand, by applying $\nabla\cdot$ to the momentum equation, we have

$$\Delta p=-\nabla v\nabla v-\nabla\cdot(\Delta\psi\nabla\psi)-\Delta\nabla\cdot\psi, \tag{2.94}$$

from which, we get the relation

$$\begin{aligned}-(\Delta\nabla p,\Delta w)&=(\Delta p,\nabla\cdot\Delta w)=-(\Delta p,\frac{1}{\mu}\Delta\nabla\cdot\psi) \\ &=\frac{1}{\mu}(\nabla v\nabla v+\nabla\cdot(\Delta\psi\nabla\psi),\Delta\nabla\cdot\psi)+\frac{1}{\mu}|\Delta\nabla\cdot\psi|_{L^2}^2.\end{aligned} \tag{2.95}$$

We estimate the right hand side of the above equation as follows:

$$\begin{aligned}&|(\nabla v\nabla v+\nabla\cdot(\Delta\psi\nabla\psi),\Delta\nabla\cdot\psi)| \\ \le{}&(|\nabla v|_{L^\infty}|\nabla v|_{L^2}|\nabla\Delta\psi|_{L^2}+|\nabla\psi|_{L^\infty}|\nabla\Delta\psi|_{L^2}^2+|\Delta\psi|_{L^4}^2|\nabla\Delta\psi|_{L^2}) \\ \le{}&C\left(|\nabla\psi|_{H^2}|\nabla v|_{L^2}^2+|\nabla v|_{L^2}|\nabla\Delta v|_{L^2}^2+(|\nabla v|_{L^2}+|\nabla\psi|_{H^2})|\nabla\Delta\psi|_{L^2}^2\right) \\ \le{}&C(|\nabla\psi|_{H^2}|\nabla v|_{L^2}^2+|\nabla v|_{L^2}|\nabla\Delta v|_{L^2}^2) \\ &+\mu^2(|\nabla v|_{L^2}+|\nabla\psi|_{H^2})(|\nabla\Delta v|_{L^2}^2+|\nabla\Delta w|_{L^2}^2)),\end{aligned}$$

To summarize, we have thus far obtained

$$\begin{aligned}-(\Delta\nabla p,\Delta w)\le{}&\frac{C}{\mu}\left(|\nabla\psi|_{H^2}|\nabla v|_{L^2}^2+\mu^2(|\nabla v|_{L^2}+|\nabla\psi|_{H^2})|\nabla\Delta w|_{L^2}^2\right. \\ &\left.+(1+\mu^2)(|\nabla v|_{L^2}+|\nabla\psi|_{H^2})|\nabla\Delta v|_{L^2}^2\right)+\frac{1}{\mu}|\Delta\nabla\cdot\psi|_{L^2}^2.\end{aligned} \tag{2.96}$$

In all these estimates, we have used the fact that $|\nabla v|_{L^\infty}\le C(|\nabla v|_{L^2}+|\nabla\Delta v|_{L^2})$ and the corresponding estimates for $|\nabla w|_{L^\infty}$ and $|\nabla\psi|_{L^\infty}$. By combining (2.90)–(2.96), we conclude

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}|\Delta w|_{L^2}^2+\frac{1}{2}\mu|\nabla\Delta w|_{L^2}^2-\frac{1}{2\mu^3}|\nabla v|_{L^2}^2-\frac{1}{\mu}|\Delta\nabla\cdot\psi|_{L^2}^2 \\ \le{}&C\Bigg((|v|_{L^2}+|\nabla w|_{L^2}+\frac{1}{\mu}|\nabla\psi|_{H^2})|\nabla v|_{L^2}^2 \\ &+(|\nabla w|_{L^2}+\mu|\nabla\psi|_{H^2}+(\frac{1}{\mu}+\mu)(|\nabla v|_{L^2}+|\nabla\psi|_{H^2})|\nabla\Delta v|_{L^2}^2 \\ &+(|v|_{L^2}+|\nabla w|_{L^2}+|\Delta w|_{L^2}+\mu|\nabla\psi|_{H^2}+\mu(|\nabla v|_{L^2}+|\nabla\psi|_{H^2}))|\nabla\Delta w|_{L^2}^2\Big) \\ \le{}&C(1+\mu+\frac{1}{\mu})(|v|_{H^2}+|\nabla\psi|_{H^2})(|\nabla v|_{L^2}^2+|\nabla\Delta v|_{L^2}^2+|\nabla\Delta w|_{L^2}^2).\end{aligned} \tag{2.97}$$

Finally, by adding up all these estimates (2.81), (2.84), (2.89) and (2.97), we find that

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}(|v|_{H^2}^2 &+ |\nabla\psi|_{H^2}^2 + |\Delta w|_{L^2}^2) \qquad (2.98)\\ &+ \frac{\mu}{2}(|\nabla v|_{L^2}^2 + |\nabla\Delta v|_{L^2}^2 + |\nabla\Delta w|_{L^2}^2) - \frac{1}{\mu}|\Delta\nabla\cdot\psi|_{L^2}^2\\ &\leq (C(1+\mu+\frac{1}{\mu})(|v|_{H^2} + |\nabla\psi|_{H^2}) - \frac{\mu}{2})(|\nabla v|_{L^2}^2 + |\nabla\Delta v|_{L^2}^2 + |\nabla\Delta w|_{L^2}^2)\\ &\quad + \frac{1}{2\mu^3}|\nabla v|_{L^2}^2.\end{aligned}$$

Therefore, if the initial data is small enough, we can find some positive T^*, such that

$$(|\nabla v|_{H^2} + |\nabla\psi|_{H^2})(t) \leq \frac{\mu}{C_0(1+\mu+\frac{1}{\mu})} \qquad (2.99)$$

for all $0 < t \leq T^*$ and C_0 large enough.

On the other hand, with the initial data satisfying $\det(I + \frac{\partial\psi_0}{\partial x}) = 1$, the first equation of (2.78) yields that $\det(I + \frac{\partial\psi(t,x)}{\partial x}) = 1$, from which we can obtain

$$\operatorname{div}\psi = \partial_1\psi_2\partial_2\psi_1 - \partial_1\psi_1\partial_2\psi_2. \qquad (2.100)$$

(2.99) together with (2.100) implies that

$$\frac{1}{\sqrt{\mu}}|\Delta\nabla\cdot\psi|_{L^2} \leq \frac{\sqrt{\mu}}{C_0(1+\mu+\frac{1}{\mu})}|\nabla\Delta\psi|_{L^2} \leq \frac{\sqrt{\mu}}{8}(|\nabla\Delta v|_{L^2} + |\nabla\Delta w|_{L^2}). \qquad (2.101)$$

Then, from (2.98), we obtain,

$$\begin{aligned}\frac{d}{dt}(|v|_{H^2}^2 + |\nabla\psi|_{H^2}^2 + |\Delta w|_{L^2}^2) + \frac{\mu}{2}(|\nabla v|_{L^2}^2 &+ |\nabla\Delta v|_{L^2}^2 \qquad (2.102)\\ &+ |\nabla\Delta w|_{L^2}^2) \leq \frac{1}{\mu^3}|\nabla v|_{L^2}^2,\end{aligned}$$

for $0 \leq t < T^*$, hence

$$\begin{aligned}&(|v|_{H^2}^2 + |\nabla\psi|_{H^2}^2 + |\Delta w|_{L^2}^2)(t) \qquad (2.103)\\ &+\frac{\mu}{2}\int_0^{T^*}\left(|\nabla v|_{L^2}^2 + |\nabla\Delta v|_{L^2}^2 + |\nabla\Delta w|_{L^2}^2\right)\,dt\\ &\leq (|v|_{H^2}^2 + |\nabla\psi|_{H^2}^2 + |\Delta w|_{L^2}^2)(0) + \int_0^{T^*}\frac{1}{\mu^3}|\nabla v|_{L^2}^2\\ &\leq (1+\frac{1}{\mu^3})(|v_0|_{H^2} + |\nabla\psi_0|_{H^2}).\end{aligned}$$

Here we have used the basic energy law in the last term of the above estimates.

In other words, if we have

$$(1+\frac{1}{\mu^3})(|v_0|_{H^2}+|\nabla\psi_0|_{H^2}) \le \frac{\mu}{C(1+\mu+\frac{1}{\mu})} \tag{2.104}$$

which is equivalent to

$$|v_0|_{H^2}+|\nabla\psi_0|_{H^2} \le \frac{\mu}{C(1+\frac{1}{\mu})^3(1+\mu+\frac{1}{\mu})}, \tag{2.105}$$

then the same estimate remains true for all the latter time with some uniform constant C independent of t. We conclude, in particular, that we can extend T^* to $+\infty$ in (2.99). Moreover, we have that:

$$\int_0^\infty (|\nabla v|_{L^2}^2+|\nabla\Delta v|_{L^2}^2+|\nabla\Delta w|_{L^2}^2)(t)\,dt \le M, \tag{2.106}$$

for a uniform constant M. This together with Theorem 2.1 imply the following global existence theorem for small initial data.

Theorem 2.2 *Let Ω is a periodic box or the whole space R^2, $k \ge 2$ be a positive integer, $\nabla\phi_0 \in H^k(\Omega)$ with det$(\nabla\phi_0)=1$ and $v_0 \in H^k(\Omega)$. Furthermore, for some large enough constant C, we assume that,*

$$|\nabla v_0|_{H^2}+|\nabla\psi_0|_{H^2} \le \frac{\mu}{C(1+\frac{1}{\mu})^3(1+\mu+\frac{1}{\mu})} \tag{2.107}$$

then the system (2.6) will have a unique global classical solution, such that,

$$|v|_{H^2}^2+|\nabla\psi|_{H^2}^2+\mu\int_0^\infty |\nabla v|_{H^2}^2\,ds \le \frac{\mu}{C(1+\mu+\frac{1}{\mu})}, \tag{2.108}$$

and (2.12) holds for $T=\infty$.

We point out that, the assumption $\det(\nabla\phi_0)=1$ is not a restriction for incompressible viscoelascity. Moreover, although (2.6) is equivalent to (1.3) only in the 2 space dimension, the proof of Theorem 2.2 actually works also for the Oldroyd systems in 3 space dimension. In particular (2.6) has a unique global smooth solution if the initial data satisfies (2.107).

Notice that the existence results we have achieve until now does not include the cases for global weak (Larey) solutions. In fact, this is still open. In the last part of this section, we give a remark that may shed some light on the difficulty of this problem. For this purpose,we consider a sequence of smooth solutions to the approximate systems. For example, let (v^ϵ, F^ϵ) be a sequence of solutions to the systems:

$$\begin{cases} F_t^\epsilon + v^\epsilon\cdot\nabla F^\epsilon = \nabla v F^\epsilon + \epsilon\Delta F^\epsilon, \\ v_t^\epsilon + v^\epsilon\cdot\nabla v^\epsilon + \nabla p^\epsilon = \mu\Delta v^\epsilon + \nabla\cdot(F^\epsilon F^{\epsilon T}), \\ \nabla\cdot v^\epsilon = 0, \end{cases} \tag{2.109}$$

with smooth initial data and appropriate boundary conditions. It is easy to prove the global existence of smooth solutions for these approximate systems in 2-D, and suitable weak solutions in 3-D, see [22]. Moreover, the energy law reads:

$$\frac{1}{2}(|v^\epsilon|_{L^2}^2 + |F^\epsilon|_{L^2}^2)(t) + \mu \int_0^t |\nabla v^\epsilon|_{L^2}^2 + \epsilon \int_0^t |\nabla F^\epsilon|_{L^2}^2 = \frac{1}{2}(|v_0^\epsilon|_{L^2}^2 + 4|\Omega|). \tag{2.110}$$

Therefore, if the initial data $v_0^\epsilon \to v_0$ in $L^2(\Omega)$, we can pass to the limit (up to a subsequence), of the sequence (v^ϵ, F^ϵ), such that, for any $T < +\infty$,

$$\begin{aligned} v^\epsilon &\rightharpoonup v \quad \text{weakly} \quad \text{in} \quad L^2(0,T;H^1), \\ F^\epsilon &\rightharpoonup F \quad \text{weak} - * \quad \text{in} \quad L^\infty(0,T;L^2). \end{aligned} \tag{2.111}$$

Using the convexity property of the matrix FF^T, we have

$$\lim_{\epsilon \to 0} F^\epsilon F^{\epsilon T} = FF^T + M, \tag{2.112}$$

where $M(x,t)$ is a positive Radom (matrix) measure.

On the other hand, since we can chose $\nabla \cdot F^\epsilon = 0$ initially, then it will be identically zero for all latter time. Thus the div-curl lemma gives the limiting equation of F, which is the same as the original one(see [21]), and the whole system becomes:

$$\begin{cases} F_t + v \cdot \nabla F = \nabla v F, \\ v_t + v \cdot \nabla v + \nabla p = \mu \Delta v + \nabla \cdot (FF^T + M), \\ \nabla \cdot v = 0. \end{cases} \tag{2.113}$$

The momentum equation is understood in the following weak sense:

$$\int_0^T \int_\Omega u_t v + \nabla u : v \otimes v - \mu \Delta u v + \nabla u : FF^T + \nabla : M \, dxdt = \int_\Omega u(x,0) v(x,0) \, dx, \tag{2.114}$$

for any smooth test function u with $\nabla \cdot u = 0$.

Lemma 2.2 *If $v \in L^2(0,T;Lip(\Omega))$, then $M = 0$.*

Sketch of proof. If $v \in L^\infty([0,T];Lip(\Omega))$, it is easy to deduce from the transport equation of F that $F \in L^\infty([0,T];L^\infty(\Omega))$, as a consequence, $M \in L^\infty(0,T;L^\infty(\Omega))$.

We can substitute v into u positions in the weak form (2.114) (by an approximate argument). After an integration by parts, one has

$$\begin{aligned} &\frac{1}{2}\int_\Omega |v|^2 \, dx + \mu \int_0^T \int_\Omega |\nabla v|^2 \, dxdt \\ &= \frac{1}{2}\int_\Omega |v_0|^2 \, dx - \int_0^T \int_\Omega \nabla v : M + \nabla v : FF^T \, dxdt \end{aligned} \tag{2.115}$$

On the other hand, from the equation for F, we have

$$\frac{1}{2}\int_{\Omega}|F|^2\,dx = 2|\Omega| + \int_0^T\int_{\Omega}\mathrm{tr}(\nabla vFF^T)\,dxdt \tag{2.116}$$

Adding up above two equations, we obtain that

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}|v|^2\,dx + \frac{1}{2}\int_{\Omega}|F|^2\,dx + \mu\int_0^T\int_{\Omega}|\nabla v|^2\,dxdt \\ &= \frac{1}{2}\int_{\Omega}|v_0|^2\,dx + 2|\Omega| - \int_0^T\int_{\Omega}\nabla v : M\,dxdt\end{aligned} \tag{2.117}$$

Subtracting (2.117) from (2.110), and

$$\lim_{\epsilon\to 0}(\frac{1}{2}\int_{\Omega}(|v^{\epsilon}|^2 - |v|^2)\,dx + \mu\int_0^T\int_{\Omega}|\nabla v^{\epsilon}|^2 - |\nabla v|^2\,dxdt) \geq 0, \tag{2.118}$$

we have

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}\mathrm{tr}M\,dx = \lim_{\epsilon\to 0}\frac{1}{2}\int_{\Omega}(|F^{\epsilon}|^2 - |F|^2)\,dx \\ &\leq \int_0^T\int_{\Omega}\nabla v : M\,dxdt \leq \int_0^T|v|_{L^\infty}\int_{\Omega}\mathrm{tr}M\,dxdt.\end{aligned} \tag{2.119}$$

The Grownwell inequality leads to the conclusion of the lemma.

3 Inviscid Case

In this section, we consider the inviscid case, that is $\mu = 0$. The system becomes:

$$\begin{cases}\phi_t + v\cdot\nabla\phi = 0,\\ v_t + v\cdot\nabla v + \nabla p = -\lambda\Delta\phi\nabla\phi,\\ \nabla\cdot v = 0.\end{cases} \tag{3.1}$$

with initial condition:

$$\phi(x,0) = \phi_0 = x + \psi_0,\ v(x,0) = v_0(x). \tag{3.2}$$

We note that this system is closely related to the hyperbolic system for the evolutions in the nonlinear elasticity studied recently by Sideris-Thomases [28]. Due to the hyperbolic nature of the system, we will consider the above problem in the entire space,and for the simplicity we shall consider the entire plane R^2. We will see the method works equally well for the 3-D or higher dimensional cases. The basic energy law governing the system now becomes :

$$\frac{d}{dt}\int_{R^2}\frac{1}{2}|v|^2 + \frac{\lambda}{2}|\nabla\phi|^2\,dx = 0. \tag{3.3}$$

We remark that, the system (3.1) possess energy laws of more general forms:

$$\frac{d}{dt}\int_{R^2}\frac{1}{2}|v|^2+\lambda(\frac{1}{2}|\nabla\phi|^2+G(\phi))\,dx=0, \tag{3.4}$$

for any scalar function $G(\cdot)$. In particular, take $G(x)=\frac{1}{\epsilon^2}(x^2-1)^2$, the Ginzburg-landau double well energy. Formally, under the scaling $\lambda=m\epsilon$, in the limit of $\epsilon\to 0$, we have that:

- $\phi\to\pm 1$, with the interface coincides with $\Gamma=\{x:\phi(x)=0\}$. In fact, ϕ almost behaviors like $\tanh(\frac{d(x,t)}{\sqrt{2}\epsilon})$ where $d(x,t)$ is the signed distance between x and Γ.
- $\int_{R^2}\lambda(\frac{1}{2}|\nabla\phi|^2+G(\phi))\,dx\to\frac{2\sqrt{2}}{3}m(\text{length of }\Gamma)$,
- $\int_{R^2}(\lambda\Delta\phi\nabla\phi,\eta)\,dx\to\int_\Gamma mH\nu\cdot\eta\,ds$, for any divergent free test function η and H and ν are the curvature and the outerward unit normal of the free interface Γ.

Hence, we can also view the system as a approximate system for the free interface problem in Euler flows.

Without lose of generality, we will assume $\lambda=1$ in the equation (4.1). If we follow the method as that presented in the proof of Theorem 2.1, we will have difficulties due to the lose of one space derivative for $\nabla\phi$. However, by taking ∇ to the ϕ equation, we can rewrite the original system as the following quasilinear first order system:

$$\begin{cases}\nabla\cdot(u_1,u_2)=0,\\ U_t+A_1(U)\nabla_1U+A_2(U)\nabla_2U=G.\end{cases} \tag{3.5}$$

with initial condition

$$U|_{x=0}=U_0. \tag{3.6}$$

Here the vector U has component u_i where, $i=1,\ldots,6$. and

$$U=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\end{pmatrix},\quad\begin{pmatrix}v_1\\v_2\\\nabla_1\phi_1\\\nabla_2\phi_1\\\nabla_1\phi_2\\\nabla_2\phi_2\end{pmatrix}=\begin{pmatrix}u_1\\u_2\\u_3+1\\u_4\\u_5\\u_6+1\end{pmatrix},\quad G=\begin{pmatrix}\nabla_1p\\\nabla_2p\\0\\0\\0\\0\end{pmatrix}$$

and the coefficient matrices are:

$$A_1(U)=\begin{pmatrix}u_1&0&u_3+1&0&u_5&0\\0&u_1&u_4&0&u_6+1&0\\u_3+1&u_4&u_1&0&0&0\\0&0&0&u_1&0&0\\u_5&u_6+1&0&0&u_1&0\\0&0&0&0&0&u_1\end{pmatrix},$$

and

$$A_2(U)=\begin{pmatrix} u_2 & 0 & 0 & u_3+1 & 0 & u_5 \\ 0 & u_2 & 0 & u_4 & 0 & u_6+1 \\ 0 & 0 & u_2 & 0 & 0 & 0 \\ u_3+1 & u_4 & 0 & u_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_2 & 0 \\ u_5 & u_6+1 & 0 & 0 & 0 & u_2 \end{pmatrix}.$$

Here we would like to point out,in particular,that the Oldroyd system (2.1) for (v, F) can also be written in the above form,and our proof below applies equally well to it. Notice the left hand side of the system (3.5) is a symmetric hyperbolic system. In order to prove the existence of solutions, we shall use an iterative scheme.

Let $U^0 = U_0 * j_1$ where $j_{1/n}$ is the smooth mollifier.

$$\Delta p^0 = \nabla v_0 \nabla v_0 - \nabla\nabla(\nabla\phi_0\nabla\phi_0). \tag{3.7}$$

Inductively, we can obtain a sequence of approximation solution (U^n, p^n) which solves

$$\begin{cases} \nabla\cdot(u_1^n, u_2^n) = 0, \\ U_t^n + A_1(U^{n-1})\nabla_1 U^n + A_2(U^{n-1})\nabla_2 U^n = G(p^{n-1}) = \begin{pmatrix} \nabla_1 p^{n-1} \\ \nabla_2 p^{n-1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\ U^n(x,0) = U_0 * j_{1/n}. \end{cases} \tag{3.8}$$

Take ∇^α to the U equation in the system, multiply by $\nabla^\alpha U^n$ and integrate over the domain,from the fact that $\nabla\cdot(u_1^n, u_2^n) = 0$, we have,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\nabla^\alpha U^n|_{L^2}^2 + (\nabla^\alpha(A_1(U^{n-1})\nabla_1 U^n), \nabla^\alpha U^n) \\ &\qquad + (\nabla^\alpha(A_2(U^{n-1})\nabla_2 U^n), \nabla^\alpha U^n) = 0. \end{aligned} \tag{3.9}$$

Now we observe that

$$\begin{aligned} &(\nabla^\alpha(A_i(U^{n-1})\nabla_i U^n), \nabla^\alpha U^n) \\ &= (\nabla^\alpha(A_i(U^{n-1})\nabla_i U^n) - A_i(U^{n-1})\nabla_i\nabla^\alpha U^n, \nabla^\alpha U^n) \\ &+(A_i(U^{n-1})\nabla_i\nabla^\alpha U^n, \nabla^\alpha U^n), \end{aligned} \tag{3.10}$$

and since $A_i(U)$ is a symmetric matrix,

$$\begin{aligned} (A_i(U^{n-1})\nabla_i\nabla^\alpha U^n\ ,\ \nabla^\alpha U^n) &= -(\nabla^\alpha U^n, \nabla_i(A_i(U^{n-1})\nabla_\alpha U^n)) \\ &= -(\nabla^\alpha U^n, \nabla_i A_i(U^{n-1})\nabla_\alpha U^n) - (\nabla^\alpha U^n, A_i(U^{n-1})\nabla_i\nabla_\alpha U^n). \end{aligned} \tag{3.11}$$

we thus deduce that

$$
\begin{aligned}
&|(\nabla^\alpha(A_i(U^{n-1})\nabla_i U^n),\nabla^\alpha U^n)| \\
&\le |\nabla^\alpha(A_i(U^{n-1})\nabla_i U^n) - A_i(U^{n-1})\nabla_i\nabla^\alpha U^n|_{L^2}|\nabla^\alpha U^n|_{L^2} \\
&+\frac{1}{2}|\nabla_i A_i(U^{n-1})|_{L^\infty}|\nabla_\alpha U^n|^2_{L^2} \\
&\le C\left(|\nabla_i U^{n-1}|_{L^\infty} + |\nabla_i A_i(U^{n-1})|_{L^\infty}\right)|\nabla_\alpha U^n|^2_{L^2}.
\end{aligned}
\tag{3.12}
$$

In other words,we have

$$
\frac{1}{2}\frac{d}{dt}|\nabla^\alpha U^n|^2_{L^2} \le C|\nabla U^{n-1}|_{L^\infty}|\nabla_\alpha U^n|^2_{L^2}
\tag{3.13}
$$

Summing up above estimates for all $|\alpha| \le k$, we get

$$
\frac{1}{2}\frac{d}{dt}|U^n|^2_{H^k} \le C|\nabla U^{n-1}|_{L^\infty}|U^n|^2_{H^k}.
\tag{3.14}
$$

Notice that $|U^n|^2_{H^k}(0) \le C$, it is standard to deduce from the above energy estimate that there is a positive time T such that for all $0 < t \le T$,

$$
|U^n|^2_{H^k}(t) \le C(T).
\tag{3.15}
$$

Le us now turn to the question of convergence of the approximate solutions. Set $W^n = U^n - U^{n-1}$ for $n \ge 1$. Then W^n, having almost the zero initial condition, will satisfy:

$$
\begin{aligned}
&W^n_t + A_1(U^{n-1})\nabla_1 W^n + (A_1(U^{n-1}) - A_1(U^{n-2}))\nabla_1 U^{n-1} \\
&+A_2(U^{n-1})\nabla_2 W^n + (A_2(U^{n-1}) - A_2(U^{n-2}))\nabla_2 U^{n-1} \\
&= \begin{pmatrix} \nabla_1(p^{n-1} - p^{n-2}) \\ \nabla_2(p^{n-1} - p^{n-2}) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
\tag{3.16}
$$

Multiply the above equation by W^n and integrate over the space variables, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|W^n|^2_{L^2} + (A_1(U^{n-1})\nabla_1 W^n, W^n) + (A_2(U^{n-1})\nabla_2 W^n, W^n) \\
&+((A_1(U^{n-1}) - A_1(U^{n-2}))\nabla_1 U^{n-1}, W^n) \\
&+((A_2(U^{n-1}) - A_2(U^{n-2}))\nabla_2 U^{n-1}, W^n) = 0.
\end{aligned}
\tag{3.17}
$$

A similar calculation as for the H^k estimates gives:

$$
\frac{1}{2}\frac{d}{dt}|W^n|^2_{L^2} \le C|\nabla U^{n-1}|_{L^\infty}(|W^n|^2_{L^2} + |W^{n-1}|_{L^2}|W^n|_{L^2}).
\tag{3.18}
$$

From the uniform bound obtained (3.15) for $|U^n|^2_{H^k}$ and the Gronwall inequality, we see that $\{U^n\}$ is a Cauchy sequence in $L^\infty(0,T;L^2(\mathbf{R}^2))$. Thus by interpolation, $\{U^n\}$ is also a Cauchy sequence in $L^\infty(0,T;H^s(\mathbf{R}^2))$, for any $s<k$. Moreover,there exists a $U(x,t)$ in $L^\infty(0,T;H^k(\mathbf{R}^2))$ such that

$$U^n \to U, \quad \text{in} \quad L^\infty(0,T;H^s(\mathbf{R}^2)). \tag{3.19}$$

Next by taking divergence of the first two components of the equation for U^n, we have

$$\Delta p^{n-1} = \sum_{i,j=1}^{2} \nabla_i\nabla_j(u_i^{n-1}u_j^n + \nabla_i\phi^{n-1}\nabla_j\phi^n), \tag{3.20}$$

where we used the divergence free property of (u_1,u_2). Then from the uniform bound (3.15), we see $\{p^{n-1}\}$ is a uniformly bounded sequence in $L^\infty(0,T;H^k(\mathbf{R}^2))$. In addition, from the convergence of U^n, $\{p^{n-1}\}$ is also a Cauchy sequence in $L^\infty(0,T;H^s(\mathbf{R}^2))$ for $s<k$. Therefore by letting n go to infinity, we get a local existence of a classical solution to the system (3.1). Similar to the above convergence proof, we can also prove the uniqueness of the smooth solution.

Theorem 3.1 *Let k be a positive integer larger than 3, $v_0 \in H^k(\mathbf{R}^2)$, $\phi_0(x)-x \in H^k(\mathbf{R}^2)$, then there exists a positive time T such that the inviscid system (3.1) has a unique solution $v(t,x) \in L^\infty(0,T;H^k(\mathbf{R}^2))$, $\nabla(\phi-x) \in L^\infty(0,T;H^k(\mathbf{R}^2))$ and $p \in L^\infty(0,T;H^k(\mathbf{R}^2))$. Moreover, if T^* is the maximal time of existence, then*

$$\int_0^{T^*} |\nabla v(\cdot,s)|_{L^\infty(\mathbf{R}^2)}\,ds = +\infty. \tag{3.21}$$

Proof. It suffices to prove the last statement (3.21). If $\int_0^{T^*} |\nabla v(\cdot,s)|_{L^\infty(\mathbf{R}^2)}\,ds < +\infty$, then by the transport equation for ϕ, we get $|\nabla\phi|_{L^\infty} < \infty$ for $0<t<T^*$. Therefore $|\nabla U|_{L^\infty} < \infty$ for $0<t<T^*$, then we can continue the the solution. This complete the proof of the Theorem.

Remarks.

a) The equation (3.5) is not a standard symmetric hyperbolic system. The fact that $\operatorname{div}(u_1,u_2)=0$ is rather crucial in our proof. Without this property, since we only have $G \in L^\infty(0,T,H^{k-1}(R^2))$, in the iteration,that will be almost impossible to get the desired estimates for U in order to make the iterative arguments to work.

b) Actually the corresponding inviscid system to (2.1) itself can be written as the following symmetric hyperbolic system:

$$\begin{cases} \nabla\cdot(v_1,v_2)=0, \\ V_t + B_1(V)\nabla_1 V + B_2(V)\nabla_2 V = G, \end{cases} \tag{3.22}$$

where the vector G is the same as that in (3.5), and

$$V = \begin{pmatrix} v_1 \\ v_2 \\ F_{11} \\ F_{12} \\ F_{21} \\ F_{22} \end{pmatrix}, \quad B_1(V) = \begin{pmatrix} v_1 & 0 & -F_{11} & -F_{12} & 0 & 0 \\ 0 & v_1 & 0 & 0 & -F_{11} & -F_{12} \\ -F_{11} & 0 & v_1 & 0 & 0 & 0 \\ -F_{12} & 0 & 0 & v_1 & 0 & 0 \\ 0 & -F_{11} & 0 & 0 & v_1 & 0 \\ 0 & -F_{12} & 0 & 0 & 0 & v_1 \end{pmatrix},$$

and

$$B_2(V) = \begin{pmatrix} v_2 & 0 & -F_{21} & -F_{22} & 0 & 0 \\ 0 & v_2 & 0 & 0 & -F_{21} & -F_{22} \\ -F_{21} & 0 & v_2 & 0 & 0 & 0 \\ -F_{22} & 0 & 0 & v_2 & 0 & 0 \\ 0 & -F_{21} & 0 & 0 & v_2 & 0 \\ 0 & -F_{22} & 0 & 0 & 0 & v_2 \end{pmatrix}.$$

Similarly in 3 space dimensional cases, the inviscid system corresponding to (2.1) can also be written as the symmetric hyperbolic system as (3.22). Moreover, by the Galilean scaling invariance of this system, Sideris and Thomases [28] recently proved the global existence of smooth solutions of a related system with sufficiently small initial data by a rather different and seemly more complicated approach. Note by the weighted energy estimates in [28], one can prove the global existence of smooth solution for small initial data only when space dimension $d \geq 3$. The corresponding result for two space dimension is open. However, with viscosity in the velocity equation, we already proved the existence result in the previous section for $d = 2$.

4 A further remark to the 3D viscous case

As mentioned in the last section, in 3 space dimensional cases, recently Sideris and Thomases [28] proved the global existence of smooth solution of a inviscid system that is closely related to (2.1) via the weighted energy estimate. Naturally their proof there could be adapted here to prove the global existence of smooth solutions to (2.1) in the 3-D cases. However, due to the specific structure of this system, we actually do not need such a complicated method here. To keep the length of this paper, we shall simply present a sketch of the proof.

Theorem 4.1 *Let* $k \geq 3$ *be an integer,* $v_0 \in H^k(\mathbf{R}^3), F_0 - I \in H^k(\mathbf{R}^3)$ *with* $\mathrm{div} v_0 =$ *and* $\mathrm{div} F = 0$. *There exists a sufficient small* $\epsilon > 0$ *such that*

$$|v_0|_{H^k} + |F_0 - I|_{H^k} \leq \epsilon. \tag{4.1}$$

Then the system (2.1) has a unique global classical solution.

Sketch of the proof. Let us linearize the system (2.1) around $(I, 0)$, and set $F = I + H$, then (2.1) can be rewritten as

$$\begin{cases} \nabla \cdot (v_1, v_2) = 0, \\ H_t = \nabla v + f_1(\nabla H, \nabla v), \\ v_t + \nabla p = \mu \Delta v + \nabla \cdot (F + F^T) + f_2(\nabla H, \nabla v). \end{cases} \tag{4.2}$$

Here $f_i, i = 1, 2$, are quadratic terms.

Note by the second equation of (2.1), we have

$$\begin{aligned} \partial_t(\nabla_l H_{ij} - \nabla_j H_{il}) + v \cdot \nabla(\nabla_l H_{ij} - \nabla_j H_{il}) + \nabla_l v \cdot \nabla H_{ij} - \nabla_j v \cdot \nabla H_{il} \\ = \nabla_k v_i(\nabla_l H_{kj} - \nabla_j H_{kl}) + \nabla_l \nabla_k v_i H_{kj} - \nabla_j \nabla_k v_i H_{kl}, \end{aligned} \tag{4.3}$$

where the convention of the summation over the repeated indices are employed.

From (4.2), one expects that $\nabla_l H_{ij} - \nabla_j H_{il} = E_{ijl}$ is a higher order term. Since $\mathrm{div} H = 0$, we have

$$\nabla \cdot (H + H^T) = \begin{pmatrix} \nabla_1 H_{11} + \nabla_2 H_{12} + \nabla_3 H_{13} \\ \nabla_1 H_{12} + \nabla_2 F_{22} + \nabla_3 H_{32} \\ \nabla_1 H_{31} + \nabla_2 H_{32} + \nabla_3 H_{33} \end{pmatrix} \triangleq A. \tag{4.4}$$

Then, one has

$$\nabla_1 A_1 = \Delta H_{11} + E, \quad \nabla_2 A_2 = \Delta H_{12} + E, \quad \nabla_3 A_3 = \Delta H_{13} + E, \tag{4.5}$$

and so on.

With the above preparation, we apply ∇_t to the second equation of (4.2) and then substitute the third equation of (4.2) into the resulting new first equation. By using (4.5), we find

$$\begin{aligned} H_{tt} &= \nabla v_t + \partial_t f_1(\nabla H, \nabla v) \\ &= -\nabla \otimes \nabla p + \mu \Delta H_t + \Delta H + E - \mu \nabla f_1(\nabla H, \nabla v) + \partial_t f_1(\nabla H, \nabla v). \end{aligned} \tag{4.6}$$

While by applying ∇_t to the third equation of (4.2) and substitute the second equation of (4.2) into the resulting new second equation, we obtain

$$v_{tt} + \nabla p_t = \mu \Delta v_t + \Delta v + \nabla \cdot f_1(\nabla H, \nabla v) + \partial_t f_2(\nabla H, \nabla v). \tag{4.7}$$

For the new system (4.6) and (4.7), we can use the method in [17] for the damped wave equation to prove the global existence of smooth solution to the system with sufficiently small initial data. In this way, we can complete the proof of the Theorem.

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