On a Micro-Macro Model for Polymeric Fluids near Equilibrium

Fang-Hua Lin* Chun Liu† Ping Zhang‡

Abstract

In this paper, we study a micro-macro model for polymeric fluid. The system involves coupling between the macroscopic momentum equation and a microscopic evolution equation describing the combined effects of the microscopic potential and thermofluctuation. We employ an energetic variation procedure to explore the relation between the macroscopic transport of the particles and the induced elastic stress due to the microscopic structure. For the initial date not far away from the equilibrium, we prove the global existence and uniqueness of classical solutions to the system.

1 Introduction

It is a classically well known fact that scales (in both spatial and temporal variables) play important roles in physical modelings. Take as an example in the description of fluids. One often uses the Schördinger type equations to model the interaction behavior or structures of its electrons. At the molecular level, one can use the Newtonian mechanics to study the molecular dynamics. At certain mesoscales, one will apply the Boltzmann type equations to describe behavior of "clouds" of molecules. At the macroscopic (continuum) level, one naturally employs the Euler or the Navier-Stokes equations to describe the fluid velocity fields, pressure and other physical quantities. At all these individual settings, the theories are clear and well accepted commonly and in most cases, represent the underlying physical properties well. One can also try to pass informations at a smaller scale to a larger one through various approaches using conservation laws for basic physical quantities. In these process (which are usually refereed to as classical mean-field theory and hydrodynamics), there seems to assume that the building elements or blocks (particles, molecule clouds, etc.) are basically of

*Courant Institute, New York University, New York, NY 10012, linf@math1.nyu.edu.

†Department of Mathematics, Pennsylvania Sate University, State College, PA 16802, liu@math.psu.edu.

‡Academy of Mathematics and System Science, Chinese Academy of Science, Beijing 100080, China zp@mail.amss.ac.cn.

same sizes or nature, so that various average process can make sense. However, in most complex (non-Newtonian) fluids,, such as polymeric fluids, one of the characteristic properties is that molecules are of very difference size (can be of more than 100 times difference), and this property does attribute to drastic different properties. In such cases, the usual average process would be difficult to yield a closure equation (or a system) at the purely macroscopic level. In most cases, the induced stress tensor in the velocity (momentum) equation involves overall integration of the entire microscopic configurational space.

In order to understand the non-Newtonian phenomena in these materials, such as shear thinning and thickening [25], in terms of the underlying molecular or microstructure of the fluids and their interaction to the fluid, we are interested in the multi-scale systems involving the detailed coupling of the molecular models for the microstructure with the macroscopic equations.

Currently there are three most commonly used approaches in the study of polymers, each one has its advantage and limitations as well. One is the direct stochastic methods, which study the distribution of the molecular behavior. The other one is through the study of molecular dynamics (MD). It gives the detailed information of the particles, but usually being very expensive in terms of computations. The third one is the direct study of the continuum models. This approach is the most widely used since it is easier to compare with the experiments and cheaper in computations. However, much physics is lost in this modeling.

To build a micro-macro model, one goes down to a microscopic scale and makes use of the kinetic theory to obtain the equation(s) governing the evolutions of the microstructure of the fluids, such as the configuration of the molecule chains in polymeric fluids. Contrary to the purely macroscopic approach where the microscopic molecules are taken into account to derive macroscopic constitutive equation (most of the time, through some simplified approximation, such as various closure assumptions) and the exact effect on the resulting system is difficult to evaluate and verify, the micro-macro approach consists of coupled systems which constantly keep explicit track of both scales.

The micro-mechanical models for polymeric liquids are usually consist of beads joined by springs or rods [5][4]. In the simplest case, the molecule configuration can be described by its end-to-end vector Q. Taking into account the elastic effect together with the thermo-fluctuation, the distribution function $\psi(x, Q, t)$ of molecule orientation Q satisfies the following Fokker-Planck equation:

$$\psi_t + De(u \cdot \nabla)\psi = \nabla_Q \cdot Dr(\nabla_Q \psi + \nabla_Q U \psi) - De \nabla_Q(\nabla u Q \psi), \tag{1.1}$$

where De is the Deborah number describing the relative time scales of the molecular relaxation in the fluid field, Dr is the dimensionless rotational diffusivity and $U(Q)$ is the potential. Notice this equation can be viewed in the Lagrangian (material) coordinate with Q as a covariant vector field transported by the fluid field u.

The above dynamics can also be written in the setting of stochastic differential equation, where the polymer-polymer interaction in the macroscopic spatial variable is neglected and also the thermal noise is assumed to be white noise in time.

Assume the fluid is incompressible, there holds

$$\operatorname{div} u = 0. \tag{1.2}$$

The momentum equation will take the form (with density being constant):

$$u_t + (u \cdot \nabla)u + \nabla p - \nu\Delta u + \nabla \cdot (\int_{R^3} Q \otimes \nabla_Q U \psi \, dQ) = 0. \tag{1.3}$$

The induced elastic stress (Kramers stress) $\int_{R^3} Q \otimes \nabla_Q U \psi \, dQ$ reflects the microscopic contribution of the polymer molecules to the overall macroscopic flow fields. This induced elastic stress can be derived by the least action principle [28]. The approach was developed in the study of liquid crystal materials [20]. This variational structure assure the following energy identity:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega [|u|^2 + De\int_{R^3} (\psi \ln \psi + U\psi)\, dQ]\, dx \\ &\qquad = -\int_\Omega [\nu|\nabla u|^2 + DeDr\int_{R^3} \psi(\nabla(\ln\psi + U))^2\, dQ]\, dx. \end{aligned} \tag{1.4}$$

The above multi-scale system can naturally be viewed as a hydrodynamic system over a fiber bundle. The current mathematical tools for kinetic equations do not seems to be suitable to treat such a system. The main difficulty lies in the transport part of the ψ equation, which is different from the standard kinetic equations [8] [7], where ψ is transported by the variable Q instead of an unknown macroscopic velocity field u. The other difficulty is the lacking of the compactness in Q variable.

The high dimension number of the independent variable of ψ gives formidable difficulty in the numerical simulations of the above system and other macro-micro models for polymeric materials. Many researchers in the engineering community perform numerical simulations of such polymer fluids through the coupling between the direct computation of the underlying stochastic equations ruling the evolution of the probability distribution function and the momentum equations. Such a hybrid strategy mixing stochastic and deterministic aspects could be advantageous in some special setting of the models [17, 19].

The alternative approach is to use the closure procedure to derive the equation in macroscopic variable only. In the simplest Hookean spring case, $U = 1/2|Q|^2$, if we take the second moment of ψ, which is the elastic stress $\tau = \int_{R^3} Q \otimes \nabla_Q U \psi \, dQ$, as a dependent variable, the system is closed in τ. Moreover, we recover the well-known Oldroyd-B equations. In other words, in this case, the second moments of the stochastic process will carry and affect all the macroscopic informations. One may hope that the closure equations will help to solve ψ. However, for more general cases, such as FENE (finite extensible nonlinearly elastic), there are no finite momentum closure systems. Even in the case of moment closure approximations, the energy law becomes inadequate to provide any compactness of τ after procedure, the previous wellposedness results of both the original problem and the closure problem are mainly the local existence [24, 18, 11]. Our results in [21] provides the global existence and uniqueness of smooth solution to the Oldroyd-B equations with small initial data, which may be important in many applications [5].

There are some analysis related to (1.1) – (1.3), mainly in the study of the incompressible Navier-Stokes equation and Boltzmann equation. In [6], Chemin proved the global existence of classical small solution to the incompressible Navier-Stokes equation by fine energy estimates. Also in [12], the authors proved a similar result by the semi-group method. However, their methods may not be applied here due to the additional term on the right hand side of (2.15). On the other hand, the pure Boltzmann equation with the perturbed initial particle distribution tends to Maxwellian exponentially, as time goes to infinity, only if the particles have the same mass and momentum as well as kinetic energy [26], due to the five-dimensional null space for the linearized collision operator. However, the global in time smooth solution close to Maxwellian to the coupled Boltzmann equations, say Vlasov-Poisson-Boltzmann or Vlasov-Maxwell-Boltzmann system, was only proved recently [15] [16]. The methods in [15] and [16] do not seem to be applicable here either, as (1.1) is actually not a standard kinetic equation due to the transport velocity here is u instead of Q as in kinetic equations. Moreover, the additional transport (stretching) term will cause many difficulties. Actually we will see in the last section that we will need to employ the methods and ideas that had been developed in [21].

Our main result , see the Theorem 4.1, concerning the system (1.1)–(1.3) is that when the initial datum is near the hydrodynamic equilibrium, where velocity is closed to zero and the probability density function is closed to a Maxwellian distribution, the system will admit a global classical solution.

The paper is written as follows: In Section 2, we formally derive the system (1.1)–(1.3), using an energetic variational approach. We focus on the special coupling of the transport of the "elastic" variable Q and the induced elastic stress. The whole system also possess a basic energy law. In Section 3, we derive several a priori estimates, including the higher order energy laws in both macroscopic x-variable derivatives and mixed derivatives. Finally, we prove the main existence theorem in Section 4.

2 A Formal Energetic Variational Derivation

The above momentum-kinetic coupling system (1.1)–(1.3) can be derived from the uniform energetic variational procedure. It is the coupling between the transport of the microscopic variable Q in the macroscopic field u and the averaged microscopic effect in the form of induced macroscopic elastic stress that give some of the interesting hydrodynamic and rheological properties of the fluid.

Throughout this paper, ∇ with no indices stands for the derivative with respect to the macroscopic variable x and ∇_Q will be the derivative with respect to the microscopic variable Q.

We first look at the dynamics in the microscopic scale. Taking a microscopic variable

$Q \in \mathbf{R}^3$, which is to be considered as the length of a molecule behaving as a 1-D spring. Given an elastic potential $U(Q)$, the dynamics of the spring (force balance) will be given as:

$$Q_{tt} + \gamma Q_t = -\nabla_Q U, \tag{2.1}$$

where γ describe the linear damping mechanism. As we are mainly concerned the case where the microscopic time scale is much faster than the macroscopic time scale. We can think that the dynamic of Q is mainly driven by the following equation:

$$\gamma Q_t = -\nabla_Q U, \tag{2.2}$$

which is just the fastest decent dynamics (gradient flow). We notice in this case, for any function $\eta(Q,t)$ of function Q, one has that

$$\gamma \eta_t + \nabla_Q U \cdot \nabla_Q \eta = 0, \tag{2.3}$$

which is just the regular transport equation.

Let us now include the thermo-fluctuation, in the form of white noise (isotropic Brownian motion), in the dynamics (2.2) of Q :

$$dQ = -\frac{1}{\gamma}\nabla_Q U dt + \sqrt{\sigma} dW_t, \tag{2.4}$$

where W_t is the regular Weiner process, and $\sigma = kT$ with k being the Boltzmann constant and T the temperature. From Ito's integration lemma [22, 13, 27], we see that the distribution function $\psi(Q,t)$ will satisfy the following Fokker-Planck equation:

$$\psi_t = \sigma \Delta_Q \psi + \frac{1}{\gamma}\nabla_Q \cdot (\nabla_Q U \psi). \tag{2.5}$$

It is obvious that this linear equation has the energy dissipation law as:

$$\frac{d}{dt}\int_{\mathbf{R}^3}\left(\sigma\psi\ln\psi + \frac{1}{\gamma}U\psi\right) dQ = -\int_{\mathbf{R}^3}\psi|\nabla_Q(\sigma\ln\psi + \frac{1}{\gamma}U)|^2\, dQ. \tag{2.6}$$

Moreover, ψ satisfies the maximal principle $\psi \geq 0$ and $\int_{\mathbf{R}^3}\psi\, dQ = 1$. Finally, the equilibrium of the equation is the pure Maxwellian:

$$\psi = C\exp\{-\frac{1}{\gamma\sigma}U\}. \tag{2.7}$$

In a macroscopic physical domain Ω, we define the particle trajectory $x(X,T)$, where X is the material coordinate, by the following the ordinary differential equation

$$x_t = u(x(X,t),t), \qquad x(X,0) = X. \tag{2.8}$$

Here $u(x,t)$ is the velocity field. We can also define the deformation tensor (composite with the push forward):

$$F(x(X,t),t) = \frac{\partial x}{\partial X}. \tag{2.9}$$

If we denote that $F_{ij} = \frac{\partial x_i}{\partial X_j}$, the tensor $F(x,t)$ satisfies the transport law:

$$F_t + u \cdot \nabla F = \nabla u F. \tag{2.10}$$

This is the consequence of chain rule. As in this paper, we only consider the incompressible case: $\det F = 1$ and $x(X,t)$ are volume preserving diffeomorphism.

Now for a macro-micro scales coupling system, we shall think that each macroscopic particle x is an ensemble of Q and also Q are transported by x. Since Q is a vector, we will regard the transport as Lie derivative [1], which includes the linear transport of the center of mass and the stretching. Hence the position dependent function $\psi(X,Q,t) = \psi(x(X,t), Fq, t)$, where q is the Lagrangian director with $Fq = Q$, and

$$\frac{D\psi}{Dt} = \psi_t + u \cdot \nabla\psi + F_t q \nabla_Q \psi = \psi_t + u \cdot \nabla\psi + \nabla u Q \cdot \nabla_Q \psi, \tag{2.11}$$

where we have used the equation (2.10). In particular, the kinetic equation becomes:

$$\psi_t + u \cdot \nabla_x \psi + \nabla_Q \cdot (\nabla u Q \psi) = \sigma \Delta_Q \psi + \frac{1}{\gamma} \nabla_Q \cdot (\nabla_Q U \psi). \tag{2.12}$$

Here we used the incompressibility $\nabla \cdot u = 0$ so that $\nabla_Q \cdot (\nabla u Q) = 0$.

For the momentum equation, we look at the competition between the macroscopic kinetic energy and the averaged effects due to the microscopic *elastic* energy. We look at the following action functional in terms of the volume preserving flow map x. Take the simple case as ρ to be 1.

$$\begin{aligned} A(x) = \int_0^T \int_\Omega \Big\{ \frac{1}{2}|x_t|^2 - \lambda \int_{\mathbf{R}^3} \big(\sigma\psi(x, Fq, t) \ln \psi(x, Fq, t) \\ + \frac{1}{\gamma} U(Q) \psi(x, Fq, t) \big) \, dQ \Big\} \, dX. \end{aligned} \tag{2.13}$$

where λ represents the ratio between the kinetic and elastic energy. Notice, there will be no variation to the potential $U(Q)$ and the volume element dQ.

The Hamilton's principle (least action principle) [9, 2] states that the force balance law (momentum equation) is equivalent to finding the critical flow maps of $A(x)$. We take a one-parameter family of volume preserving flow maps x^ϵ, such that $\frac{d}{d\epsilon} x^\epsilon|_{\epsilon=0} = y$, hence $\nabla \cdot y = 0$.

$$\begin{aligned} \frac{d}{d\epsilon} A(x)|_{\epsilon=0} &= \int_0^T \int_\Omega \Big\{ x_t y_t - \lambda \int_{\mathbf{R}^3} \Big(\sigma(1 + \ln\psi) + \frac{1}{\gamma} U) \nabla\psi y \\ &\quad + (\sigma(1+\ln\psi) + \frac{1}{\gamma} U) \nabla_Q \psi \nabla_X y q \Big) \, dQ \Big\} \, dX \\ &= \int_0^T \int_\Omega \Big\{ x_t y_t - \lambda \int_{\mathbf{R}^3} \Big(-(\sigma\psi \ln\psi + \frac{1}{\gamma} U\psi) \nabla \cdot y \\ &\quad + (-\sigma\psi \ln\psi \nabla_Q \cdot (\nabla y Q) - \frac{1}{\gamma} \nabla_Q U \psi \nabla y Q) \Big) \, dQ \Big\} \, dX \\ &= \int_0^T \int_\Omega \Big(x_t y_t - \frac{\lambda}{\gamma} \int_\Omega \nabla_Q U \psi \nabla y Q \, dQ \Big) \, dx. \end{aligned} \tag{2.14}$$

After postulating the viscosity (or including randomness in the last variation [23]), we obtain the following momentum equation:

$$u_t + u \cdot \nabla u + \nabla p = \mu \Delta u - \frac{\lambda}{\gamma} \nabla \cdot (\int_{\mathbf{R}^3} \nabla_Q U \otimes Q\psi \, dQ), \tag{2.15}$$

where the pressure p is the Lagrange multiplier corresponding to the constraint $\nabla \cdot u = 0$.

The system (2.12) (2.15), together with the incompressibility $\nabla \cdot u = 0$ can be viewed as a two scale system defined on the fiber bundle $\Omega \times \mathbf{R}^3$. It will still satisfy the energy identity:

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \left(\frac{1}{2}|u|^2 + \lambda \int_{\mathbf{R}^3} (\sigma \psi \ln \psi + \frac{1}{\gamma} U\psi) \, dQ \right) \, dx \\ &= - \int_\Omega (\mu |\nabla u|^2 + \lambda \int_{\mathbf{R}^3} \psi |\nabla_Q (\sigma \ln \psi + \frac{1}{\gamma} U)|^2 \, dQ) \, dx. \end{aligned} \tag{2.16}$$

This completes the formal derivation of the micro-macro system (1.1)–(1.3) modeling the polymeric fluids.

Now let us introduce some basic notation that will be used throughout the rest of the paper. Without lose of generality, from now on, except σ and μ, we assume that all the other constants in the system (2.12) (2.15) equal 1. Moreover after renormalization, we will assume that $\int_{\mathbf{R}^3} \exp\{-U\} \, dQ = 1$. Also, to avoid additional technicalities, we assume that:

$$\begin{aligned} &|Q| \le C(|\nabla_Q U| + 1), \quad \Delta U \le C + \delta |\nabla_Q U|^2, \text{ where } \delta < 1, \\ &\int_{\mathbf{R}^3} |\nabla_Q U|^2 e^{-U} \, dQ \le C, \quad \int_{\mathbf{R}^3} |Q|^4 e^{-U} \, dQ \le C, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} &|\nabla_Q^k (Q \nabla_Q U) \le C(|Q||\nabla_Q U| + 1), \quad \int_{\mathbf{R}^3} |\nabla_Q^k (Q \nabla_Q U e^{-\frac{U}{2}})|^2 \, dQ \le C, \\ &|\nabla_Q^k (\Delta_Q U - \frac{|\nabla_Q U|^2}{2})| \le C(1 + |\nabla_Q U|^2), \end{aligned} \tag{2.18}$$

where the positive integer $1 \le k \le s$, which will be fixed in the later sections.

Note that the first inequality of (2.17) is equivalent to say that $|Q| \le C|\nabla_Q U|$ hold for $|Q|$ large enough. For simplicity, sometimes we just assume that$|Q| \le C|\nabla_Q U|$ hold on the whole space and take $\delta = \frac{1}{2}$ in the subsequence, which avoid some unnecessary complications but will not alter any of our proofs.

Also we make the specification that $\Omega = \mathbb{T}^3$, or $\mathbf{R}^3$. For any give function ϕ, we denote

$$|\phi|_{L^p} = \int_\Omega \int_{\mathbf{R}^3} |\phi|^p \, dQ dx, \quad |\phi|_{H^s} = \int_\Omega \int_{\mathbf{R}^3} \sum_{|\alpha| \le s} |\nabla^\alpha \phi|^2 \, dQ dx,$$

and $\|\phi\|_{H^s}$ the common Sobolev norm of ϕ. We shall use the convention (f,g) to stand for both the inner product on Ω, $\int_\Omega fg\,dx$, and on $\Omega\times\mathbf{R}^3$, $\int_\Omega\int_{\mathbf{R}^3} fg\,dQdx$. And we will denote ∇^s be any of ∇^α, where α is any multiindice with $|\alpha|=s$. And similar notation for ∇_Q^s.

3 *A priori* estimates

Motivated by [26], and [15], [16], we define the perturbation of ψ around the pure Maxwellian stationary solution:

$$\psi = e^{-U} + e^{-U/2}f \triangleq M + \sqrt{M}f. \tag{3.1}$$

Note that we assume that $\int_{\mathbf{R}^3}\psi_0(x,Q)\,dQ = 1$, then formally from (2.12), there holds $\int_{\mathbf{R}^3}\psi(t,x,Q)\,dQ=1$, which together with (3.1) yields

$$\int_{\mathbf{R}^3}\sqrt{M}f\,dQ = 0. \tag{3.2}$$

Plugging (3.1) to (2.12) and (2.15) respectively, we get the new system for (u,f):

$$\begin{cases} f_t + u\cdot\nabla f + \nabla uQ\cdot\nabla_Q f - \sigma(\Delta_Q f + \frac{\Delta_Q U}{2}f - \frac{|\nabla_Q U|^2}{4}f) \\ = \sqrt{M}\nabla uQ\cdot\nabla_Q U + \frac{1}{2}\nabla uQ\cdot\nabla_Q U f, \quad (x,Q)\in\Omega\times\mathbf{R}^3, \\ u_t + u\cdot\nabla u + \nabla p = \mu\Delta u + \nabla\cdot(\int_{\mathbf{R}^3}\nabla_Q U\otimes Q\sqrt{M}f\,dQ), \quad x\in\Omega. \end{cases} \tag{3.3}$$

In what follows, we shall consider the global existence of classical small solution for (3.3) together with the initial condition

$$f|_{t=0} = f_0, \qquad u|_{t=0} = u_0, \quad \text{for} \quad (x,Q)\in\Omega\times\mathbf{R}^3, \tag{3.4}$$

where f_0 satisfies (3.2), $\mathrm{div}u_0 = 0$, and f_0, u_0 are sufficiently small.

Now let us first outline the procedure of proof of the a priori estimates in the next two subsections. The a priori estimates can be separated into two parts in the following subsections respectively, the pure x variable one and the mixed derivative one. We use the following step-by-step procedure to prove the first part:

- The usual method for the higher order energy estimates will lead to (3.5) and (3.7). Notice that the worst term $\int_\Omega\int_{\mathbf{R}^3}\nabla_Q U\otimes Q\sqrt{M}\nabla^s f\nabla\nabla^s u\,dQdx$ will be canceled if we add these two equations together.
- To proceed, we use a pointwise (in x variable) inequality (Lemma 3.1). later on, we will apply this inequality for the interpolating estimates.
- Next we derive (3.25) and (3.26) where we use interpolation to bound some of the bad terms.

- The above Gronwall type estimates are still not closed as the right hand sides involves additional terms which have to be bounded through mixed derivative estimates.

- Moreover, the right hand side of (3.25) and (3.26) also involve higher order moment terms for the distribution in the kinetic variable Q, most notably the term $|Qf|_{H^\gamma}$, for $0 \le \gamma \le 4$. These terms are handled by certain weighted dissipation along with the weighted Poincare inequality (Lemma 3.2).

The mixed derivative estimates follows almost the exact same steps. Finally we combine all these estimate to prove the global existence of the classical solutions of (3.3)–(3.4). At this moment we need to use the technique we had developed [21] in studying the Oldroyd models. Namely we have to use the basic energy law (2.16).

3.1 Higher Order Energy Estimates in x Variable

In this subsection, we will focus on the estimates for $x-$derivatives of the unknown variables. Let $s \ge 7$ be a positive integer. We apply ∇^s to the velocity equation in (3.3), then multiply the resulting equation by $\nabla^s u$ and integrate over Ω to get,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla^s u|_{L^2}^2 + \mu|\nabla\nabla^s u|_{L^2}^2 &= -(\nabla^s(u\cdot\nabla u), \nabla^s u) \\ &\quad - \int_\Omega\int_{\mathbf{R}^3} \nabla_Q U \otimes Q\sqrt{M}\nabla^s f \nabla\nabla^s u \, dQdx, \end{aligned} \tag{3.5}$$

while as $\mathrm{div} u = 0$, by integration by parts,

$$|(\nabla^s(u\cdot\nabla u), \nabla^s u)| = |(\nabla^s(u\otimes u), \nabla\nabla^s u)| \le C|u|_{H^s}|\nabla u|_{H^s}^2. \tag{3.6}$$

A similar process for the microscopic equation in (3.3) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\nabla^s f|_{L^2}^2 + \sigma\left(|\nabla_Q\nabla_x^s f|_{L^2}^2 - \frac{1}{2}(\Delta_Q U\nabla^s f, \nabla^s f) + \frac{1}{4}|\nabla_Q U\nabla^s f|_{L^2}^2\right) \\ &\qquad = -(\nabla^s(u\cdot\nabla f), \nabla^s f) - (\nabla^s(\nabla u Q\nabla_Q f), \nabla^s f) \\ &\qquad + \int_\Omega\int_{\mathbf{R}^3} \nabla_Q U \otimes Q\sqrt{M}\nabla^s f\nabla\nabla^s u\, dQdx + \frac{1}{2}(\nabla^s(\nabla u Q\nabla_Q U f), \nabla^s f).\end{aligned} \tag{3.7}$$

Notice that the first term in the last line of (3.7) cancels the last term in (3.5). By integration by parts, we can get

$$-\frac{1}{2}(\Delta_Q U\nabla^s f, \nabla^s f) = (\nabla_Q U\nabla_x^s f, \nabla_Q\nabla_x^s f).$$

Thus the dissipation due to the Fokker-Planck kinetic dynamics is given by:

$$\begin{aligned}&|\nabla_Q\nabla^s f|_{L^2}^2 - \frac{1}{2}(\Delta_Q U\nabla_x^s f, \nabla_x^s f) + \frac{1}{4}|\nabla_Q U\nabla^s f|_{L^2}^2 \\ &= |\nabla_Q\nabla^s f + \frac{1}{2}\nabla_Q U\nabla^s f|_{L^2}^2.\end{aligned} \tag{3.8}$$

To proceed, we will need the following lemma, which is a direct consequence of Sobolev imbedding theorem that $W^{4,1}(\Omega)$ can be imbedded in $L^\infty(\Omega)$.

Lemma 3.1 *Let $f(x,Q) \in \mathcal{S}(\Omega \times \mathbf{R}^3)$, there exist a uniform constant C such that*

$$\sup_{x\in\Omega} \int_{\mathbf{R}^3} f^2 \, dQ \le C \sum_{|\gamma|\le 4} \int_\Omega \int_{\mathbf{R}^3} |\nabla^\gamma f^2| \, dQ \, dx \le C|f|^2_{H^4}. \tag{3.9}$$

With the above lemma, we can estimate the remaining terms in (3.7) separately. First of all, since $\mathrm{div} u = 0$, we have the following estimate:

$$\begin{aligned} |(\nabla^s(u\cdot\nabla f), \nabla^s f)| &= |(\nabla^s(u\cdot\nabla f) - u\cdot\nabla\nabla^s f, \nabla^s f)| \\ &\le \sum_{d_1+d_2=s, d_1\ge 1} |(\nabla^{d_1} u \cdot \nabla\nabla^{d_2} f, \nabla^s f)|. \end{aligned} \tag{3.10}$$

In order to apply the Lemma 3.1 to estimate the right hand side of (3.10), we split the sum in (3.10) into two parts. The first part consists of terms with $1 \le d_1 \le 5$, since $s \ge 7$, we have

$$\begin{aligned} |(\nabla^{d_1} u \cdot \nabla\nabla^{d_2} f, \nabla^s f)| &\le C|\nabla^{d_1} u|_{L^\infty} |\nabla^{d_2+1} f|_{L^2} |\nabla^s f|_{L^2} \\ &\le C|u|_{H^s} |\nabla^{d_2+1} f|_{L^2} |\nabla^s f|_{L^2}. \end{aligned} \tag{3.11}$$

The second part consists those terms with $d_1 > 5$. By Lemma 3.1 we obtain

$$\begin{aligned} |(\nabla^{d_1} u \nabla\nabla^{d_2} f, \nabla^s f)| &\le \sup_{x\in\Omega}\Big(\int_{\mathbf{R}^3} |\nabla^{d_2+1} f|^2 \, dQ\Big)^{1/2} |\nabla^{d_1} u|_{L^2} |\nabla^s f|_{L^2} \\ &\le C \sum_{|\gamma|\le 5} |\nabla^{\gamma+d_2} f|_{L^2} |\nabla^{d_1} u|_{L^2} |\nabla^s f|_{L^2} \\ &\le C|f|_{H^s} |u|_{H^s} |\nabla^s f|_{L^2}. \end{aligned} \tag{3.12}$$

Here $\gamma + d_2 \le 5 + d_2 < d_1 + d_2 = s$.

Summing up (3.10)-(3.12), we obtain

$$|(\nabla^s(u\cdot\nabla f), \nabla^s f)| \le C|u|_{H^s} |f|_{H^s} |\nabla^s f|_{L^2}, \tag{3.13}$$

as $d_1 \ge 1, d_2 + 1 \le s$.

On the other hand, $\nabla \cdot u = 0$ implies that $\nabla_Q \cdot (\nabla u Q) = 0$, from which we obtain:

$$\begin{aligned} |(\nabla^s(\nabla u Q \cdot \nabla_Q f), \nabla^s f)| &= |(\nabla^s(\nabla u Q\cdot\nabla_Q f) - \nabla u Q \cdot \nabla_Q \nabla^s f, \nabla^s f)| \\ &\le \sum_{d_1+d_2=s, d_1\ge 1} |(\nabla^{d_1}\nabla u Q \cdot \nabla^{d_2} f, \nabla_Q \nabla^s f)|. \end{aligned} \tag{3.14}$$

We can now estimate the term corresponding to $d_1 = s$ in the above by

$$\begin{aligned} |(\nabla^s \nabla u Q \cdot \nabla_Q f, \nabla^s f)| &\le \sup_{x\in\Omega}\Big(\int_{\mathbf{R}^3} |\nabla_Q f|^2 \, dQ\Big)^{1/2} |\nabla u|_{H^s} |Q\nabla^s f|_{L^2} \\ &\le C|\nabla_Q f|_{H^4} |\nabla u|_{H^s} |\nabla_Q U \nabla^s f|_{L^2}. \end{aligned} \tag{3.15}$$

where in the last step, we used the minor technical assumption (2.17) on U. The right hand side of (3.15) will have to be combined with the energy laws for mixed derivatives, and will be handled in the next subsection, the Subsection 3.2.

For $1 \le d_1 \le s-1$, we use the decomposition

$$\nabla_Q \nabla^s f = (\nabla_Q \nabla^s f + \frac{1}{2}\nabla_Q U \nabla^s f) - \frac{1}{2}\nabla_Q U \nabla^s f.$$

Similar to the estimate of (3.13), we separate the summand in (3.14) into the following two cases: The first case deals with those terms where $d_1 \le 4$, we observe that

$$\begin{aligned}
&|(\nabla^{d_1}\nabla u Q \nabla^{d_2} f, \nabla_Q \nabla^s f)| \\
&\le |\nabla^{d_1+1} u|_{L^\infty} |Q\nabla^{d_2} f|_{L^2} \left(|\nabla_Q \nabla^s f + \frac{1}{2}\nabla_Q U \nabla^s f|_{L^2} + \frac{1}{2}|\nabla_Q U \nabla^s|_{L^2} \right) \\
&\le C|u|_{H^s} |\nabla_Q U \nabla^{d_2} f|_{L^2} \left(|\nabla_Q \nabla^s f + \frac{1}{2}\nabla_Q U \nabla^s f|_{L^2} + \frac{1}{2}|\nabla_Q U \nabla^s|_{L^2} \right). \qquad (3.16)
\end{aligned}$$

Here we used again the technical assumption (2.17) on U.

The second case is for the terms where $4 < d_1 \le s-1$. By Lemma 3.1, we have

$$\begin{aligned}
&|(\nabla^{d_1}\nabla u Q \nabla^{d_2} f, \nabla_Q \nabla^s f)| \\
&\le \sup_{x\in\Omega}\Big(\int_{\mathbf{R}^3} |Q\nabla^{d_2} f|^2 \, dQ\Big)^{1/2} |\nabla^{d_1+1} u|_{L^2} \left(|\nabla_Q \nabla^s f + \frac{1}{2}\nabla_Q U \nabla^s f|_{L^2} + \frac{1}{2}|\nabla_Q U \nabla^s f|_{L^2} \right) \\
&\le C \sum_{|\gamma|\le 4} |Q\nabla_x^{d_2+\gamma} f|_{L^2} |u|_{H^s} \left(|\nabla_Q \nabla^s f + \frac{1}{2}\nabla_Q U \nabla^s f|_{L^2} + \frac{1}{2}|\nabla_Q U \nabla^s f|_{L^2} \right) \\
&\le C|u|_{H^s} |\nabla_Q U f|_{H^s} \left(|\nabla_Q \nabla_x^s f + \frac{1}{2}|\nabla_Q U \nabla_x^s f|_{L^2} + \frac{1}{2}|\nabla_Q U \nabla^s f|_{L^2} \right). \qquad (3.17)
\end{aligned}$$

Here we used the facts that $4 + d_2 \le s$ and $|Q| \le C|\nabla_Q U|$.

Combining (3.14) through (3.17), we arrive at

$$\begin{aligned}
|(\nabla^s(\nabla u Q \nabla_Q f), \nabla^s f)| \le C\{&|\nabla_Q f|_{H^4} |\nabla u|_{H^s} |\nabla_Q U \nabla^s f|_{L^2} \qquad (3.18) \\
&+|u|_{H^s} |\nabla_Q U f|_{H^s} (|\nabla_Q \nabla^s f + \frac{1}{2}\nabla_Q U \nabla^s f|_{L^2} \qquad (3.19) \\
&+|\nabla_Q U \nabla^s f|_{L^2})\}.
\end{aligned}$$

Next let us turn to the estimate of the last remaining term in (3.7). By Lebnitz's formula, there holds

$$\begin{aligned}
|(\nabla^s(\nabla u Q \nabla_Q U f), \nabla^s f)| \le &\sum_{d_1+d_2=s, d_1\le s-1} |(\nabla^{d_1}\nabla u Q \nabla_Q U \nabla^{d_2} f, \nabla^s f)| \\
&+|(\nabla^s \nabla u Q \nabla_Q U f, \nabla^s f)|. \qquad (3.20)
\end{aligned}$$

For the cases $d_1 \le 4$, since $s \ge 7$, we have

$$\begin{aligned}
|(\nabla^{d_1}\nabla u Q \nabla_Q U \nabla^{d_2} f, \nabla^s f)| &\le |\nabla^{d_1+1} u|_{L^\infty} |Q\nabla^{d_2} f|_{L^2} |\nabla_Q U \nabla^s f|_{L^2} \\
&\le C|u|_{H^s} |\nabla_Q U \nabla^{d_2} f|_{L^2} |\nabla_Q U \nabla_x^s f|_{L^2}. \qquad (3.21)
\end{aligned}$$

For $s-1\geq d_1\geq 5$, since $|\gamma|\leq 4$, $\gamma+d_2\leq s$,

$$
\begin{aligned}
|(\nabla^{d_1}\nabla uQ\nabla_Q U\nabla^{d_2}f,\nabla^s f)| &\leq \sup_{x\in\Omega}(\int_{\mathbf{R}^3}|\nabla_Q U\nabla^{d_2}f|^2\,dQ)^{1/2}|\nabla^{d_1+1}u|_{L^2}|Q\nabla^s f|_{L^2}\\
&\leq C\sum_{\gamma\leq 4}|\nabla_Q U\nabla^{\gamma+d_2}f|_{L^2}|\nabla^{d_1+1}u|_{L^2}|\nabla_Q U\nabla^s f|_{L^2}\\
&\leq C|u|_{H^s}|\nabla_Q Uf|_{H^s}|\nabla_Q U\nabla^s f|_{L^2}. \qquad (3.22)
\end{aligned}
$$

Finally, when $d_1=s$, we have

$$
\begin{aligned}
|(\nabla^s\nabla uQ\nabla_Q Uf,\nabla^s f)| &\leq \sup_{x\in\Omega}(\int_{\mathbf{R}^3}|Qf|^2\,dQ)^{1/2}|\nabla u|_{H^s}|\nabla_Q U\nabla^s f|_{L^2}\\
&\leq |Qf|_{H^4}|\nabla u|_{H^s}|\nabla_Q U\nabla^s f|_{L^2}. \qquad (3.23)
\end{aligned}
$$

Combining (3.20)–(3.23), we obtain that

$$
|(\nabla^s(\nabla uQ\nabla_Q Uf),\nabla^s f)|\leq C(|u|_{H^s}|\nabla_Q Uf|_{H^s}+|Qf|_{H^4}|\nabla u|_{H^s})|\nabla_Q U\nabla^s f|_{L^2}. \qquad (3.24)
$$

Now let us pick up all the x derivative estimates together. By combining (3.5) and (3.7), and using (3.6), (3.8), (3.13), (3.18) and (3.24), we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\nabla^s u|_{L^2}^2+|\nabla^s f|_{L^2}^2)+\mu|\nabla^s\nabla u|_{L^2}^2+\sigma|\nabla_Q\nabla^s f+\frac{1}{2}\nabla_Q U\nabla^s f|_{L^2}^2\\
&\leq C\Big(|u|_{H^s}|\nabla u|_{H^s}^2+|f|_{H^s}|u|_{H^s}|\nabla^s f|_{L^2}+(|\nabla_Q f|_{H^4}+|Qf|_{H^4})|\nabla u|_{H^s}|\nabla_Q U\nabla^s f|_{L^2}\\
&\quad+|u|_{H^s}|\nabla_Q Uf|_{H^s}(|\nabla_Q\nabla^s f+\frac{1}{2}\nabla_Q U\nabla^s f|_{L^2}+|\nabla_Q U\nabla^s f|_{L^2})\Big). \qquad (3.25)
\end{aligned}
$$

Note that for $0\leq k\leq s-1$, terms do not have the highest order derivatives like those in the left hand side of (3.15) and (3.23) and hence can be estimated through (3.16) – (3.22), therefore by a similar proof of (3.25), we obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\nabla^k u|_{L^2}^2+|\nabla^k f|_{L^2}^2)+\mu|\nabla\nabla^k u|_{L^2}^2+\sigma|\nabla_Q\nabla^k f+\frac{1}{2}\nabla_Q U\nabla^k f|_{L^2}^2 \qquad (3.26)\\
&\leq C\Big(|u|_{H^k}|\nabla u|_{H^s}|\nabla u|_{H^k}+|f|_{H^s}|u|_{H^s}|\nabla^k f|_{L^2}+|u|_{H^s}|\nabla_Q Uf|_{H^s}(|\nabla_Q U\nabla^k f\\
&\quad+\frac{1}{2}\nabla_Q U\nabla^k f|_{L^2}+|\nabla_Q U\nabla^k f|_{L^2})\Big).
\end{aligned}
$$

Here we used the fact $(u\cdot\nabla u,u)=0$ as $\operatorname{div}u=0$.

In order to estimate the term $|Qf|_{H^4}$ in (3.25) and (3.26), we apply ∇^γ , for $0\leq\gamma\leq 4$, to the microscopic equation (3.3), and take the L^2 inner product of the resulting equation with $|Q|^2\nabla^\gamma f$ to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||Q|\nabla^\gamma f|_{L^2}^2+\sigma((-\Delta_Q-\frac{\Delta_Q U}{2}+\frac{|\nabla_Q U|^2}{4})\nabla^\gamma f,|Q|^2\nabla^\gamma f)\\
&=-(\nabla^\gamma(u\cdot\nabla f),|Q|^2\nabla^\gamma f)-(\nabla^\gamma(\nabla uQ\cdot\nabla_Q f),|Q|^2\nabla^\gamma f) \qquad (3.27)\\
&\quad+(\sqrt{M}\nabla^\gamma\nabla uQ\cdot\nabla_Q U,|Q|^2\nabla^\gamma f)+\frac{1}{2}(\nabla^\gamma(\nabla uQ\cdot\nabla_Q Uf),|Q|^2\nabla^\gamma f).
\end{aligned}
$$

In what follows, we will estimate the above terms separately. Firstly similar to the proof of (3.8), by integration by parts, we obtain

$$
\begin{aligned}
&((-\Delta_Q - \frac{\Delta_Q U}{2} + \frac{|\nabla_Q U|^2}{4})\nabla^\gamma f, |Q|^2\nabla^\gamma f) \qquad (3.28)\\
&\quad = ||Q|(\nabla_Q\nabla^\gamma f + \frac{1}{2}\nabla_Q U\nabla^\gamma f)|_{L^2}^2 - 3|\nabla^\gamma f|_{L^2} + \int_\Omega\int_{\mathbf{R}^3}\nabla_Q U\cdot Q|\nabla^\gamma f|^2\,dQ\,dx.
\end{aligned}
$$

While as $s \geq 7$, by a similar proof of (3.13), there holds

$$
\begin{aligned}
|(\nabla^\gamma(u\cdot\nabla f), |Q|^2\nabla^\gamma f)| &\leq \sum_{\gamma_1+\gamma_2=\gamma,\gamma_1\geq 1}|(\nabla^{\gamma_1}u\cdot\nabla^{\gamma_2+1}f, |Q|^2\nabla^\gamma f)|\\
&\leq C|f|_{H^\gamma}|\nabla u|_{H^s}||Q|^2\nabla^\gamma f|_{L^2}. \qquad (3.29)
\end{aligned}
$$

Following the decomposition in (3.14), we can estimate the last term of the second line of (3.27) as follows:

$$
\begin{aligned}
|(\nabla^\gamma(\nabla u Q\cdot\nabla_Q f)\ &,\ |Q|^2\nabla^\gamma f)|\\
&\leq 2|(\nabla^\gamma(\nabla u Q f), Q\nabla^\gamma f)| + |(\nabla^\gamma(\nabla u Q f), |Q|^2\nabla_Q\nabla^\gamma f)|\\
&\leq C|u|_{H^s}\Big(|Qf|_{H^\gamma}|Q\nabla^\gamma f|_{L^2} + ||Q|^2 f|_{H^\gamma}(|Q(\nabla^\gamma\nabla_Q f\\
&\qquad + \frac{1}{2}\nabla_Q U\nabla^\gamma f)|_{L^2} + |Q\nabla_Q U\nabla^\gamma f|_{L^2})\Big). \qquad (3.30)
\end{aligned}
$$

Finally, it is rather easy to see that

$$
|(\sqrt{M}\nabla^\gamma\nabla u Q\cdot\nabla_Q U, |Q|^2\nabla^\gamma f)| \leq |\nabla u|_{H^\gamma}(\int_{\mathbf{R}^3}(M|Q|^4)\,dQ)^{\frac{1}{2}}|Q\nabla_Q U\nabla^\gamma f|_{L^2}, \qquad (3.31)
$$

$$
|(\nabla^\gamma(\nabla u Q\cdot\nabla_Q U f), |Q|^2\nabla^\gamma f)| \leq |u|_{H^s}|Q\cdot\nabla_Q U f|_{H^\gamma}||Q|^2\nabla^\gamma f|_{L^2}. \qquad (3.32)
$$

By summing up (3.27) through (3.32), and making the summation for γ from 0 to 4. we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||Q|f|_{H^4} + \sigma||Q|(\nabla_Q f + \frac{1}{2}\nabla_Q U f)|_{H^4}^2 \qquad (3.33)\\
&\leq C\Bigg(\sigma[|f|_{H^4} + \sum_{\gamma\leq 4}\int_\Omega\int_{\mathbf{R}^3}|\nabla_Q U\cdot Q||\nabla^\gamma f|^2\,dQ\,dx] + |f|_{H^4}|\nabla u|_{H^s}||Q|^2 f|_{H^4}\\
&\quad + |u|_{H^s}[|Qf|_{H^4}^2 + ||Q|^2 f|_{H^4}(|Q(\nabla_Q f + \frac{1}{2}\nabla_Q U f)|_{H^4} + |Q\cdot\nabla_Q U f|_{H^4})]\\
&\quad + |\nabla u|_{H^4}|Q\cdot\nabla_Q U f|_{H^4}\Bigg).
\end{aligned}
$$

To use the dissipation term to control the weighted L^2 estimate of f, we need the following Lemma.

Lemma 3.2 *Let f satisfy $\int_{\mathbf{R}^3} f\sqrt{M}\,dQ = 0$, and U satisfies (2.17). Then, there holds*

$$|f|_{L^2} \leq C|\nabla_Q f + \frac{1}{2}\nabla_Q U f|_{L^2}, \tag{3.34}$$

$$|\nabla_Q U f|_{L^2} \leq C|\nabla_Q f + \frac{1}{2}\nabla_Q U f|_{L^2}, \tag{3.35}$$

$$|\nabla_Q U Q f|_{L^2}^2 \leq C|(1+|Q|^2)^{\frac{1}{2}}(\nabla_Q f + \frac{1}{2}\nabla_Q U f)|_{L^2}^2. \tag{3.36}$$

Proof. Let us set $g = e^{\frac{1}{2}U} f$, then (3.34) (3.35) are respectively equivalent to

$$\int_{\mathbf{R}^3} g^2 e^{-U}\,dQ \leq C\int_{\mathbf{R}^3} |\nabla_Q g|^2 e^{-U}\,dQ, \tag{3.37}$$

$$\int_\Omega\int_{\mathbf{R}^3} |\nabla_Q U|^2 g^2 e^{-U}\,dQ\,dx \leq C\int_\Omega\int_{\mathbf{R}^3} |\nabla_Q g|^2 e^{-U}\,dQ\,dx. \tag{3.38}$$

However as $\int_{\mathbf{R}^3} f\sqrt{M}\,dQ = 0$ implies that $\int_{\mathbf{R}^3} g e^{-U}\,dQ = 0$, then Poincare inequality [14] implies (3.37).

On the other hand, by integration by parts, we have

$$\begin{aligned}\int_{\mathbf{R}^3} |\nabla_Q U|^2 g^2 e^{-U}\,dQ &= \int_{\mathbf{R}^3} (\Delta U g^2 + 2\nabla_Q U g \nabla_Q g)E^{-U}\,dQ \\ &\leq \int_{\mathbf{R}^3} (C + \frac{3}{4}|\nabla_Q U|^2) g^2 e^{-U}\,dQ + 4\int_{\mathbf{R}^3} |\nabla_Q g|^2 e^{-U}\,dQ,\end{aligned} \tag{3.39}$$

where we used the assumption (2.17) that $\Delta U \leq C + \frac{1}{2}|\nabla_Q U|^2$. Combining (3.39) with (3.37), we obtain

$$\int_{\mathbf{R}^3} |\nabla_Q U|^2 g^2 e^{-U}\,dQ \leq C\int_{\mathbf{R}^3} |\nabla_Q g|^2 e^{-U}\,dQ.$$

Integrating the above inequality over Ω, we prove (3.38).

With (3.35), we can now prove (3.36). Actually by the definition of g, one has

$$\begin{aligned}&\int_\Omega\int_{\mathbf{R}^3} |Q|^2|\nabla_Q U|^2 f^2\,dQ\,dx = \int_\Omega\int_{\mathbf{R}^3} |\nabla_Q U|^2|Q|^2 g^2 e^{-U}\,dQ\,dx \\ &\leq 2\int_\Omega\int_{\mathbf{R}^3} |\nabla_Q U|^2 |Qg - \int_{\mathbf{R}^3} Qg e^{-U}\,dQ|^2 e^{-U}\,dQ\,dx \\ &\quad +2\int_{\mathbf{R}^3} |\nabla_Q U|^2 e^{-U}\,dQ \int_\Omega |\int_{\mathbf{R}^3} Qg e^{-U}\,dQ|^2\,dx,\end{aligned} \tag{3.40}$$

On the other hand, by (3.37) and (3.38), one has

$$\begin{aligned}\int_\Omega\int_{\mathbf{R}^3} |\nabla_Q U|^2 |Qg - \int_{\mathbf{R}^3} Qg e^{-U}\,dQ|^2 e^{-U}\,dQ\,dx &\leq \int_\Omega\int_{\mathbf{R}^3} |\nabla_Q (Qg)|^2 e^{-U}\,dQ\,dx \\ &\leq C\int_\Omega\int_{\mathbf{R}^3} (1+|Q|^2)|\nabla_Q g|^2\,dQ\,dx,\end{aligned} \tag{3.41}$$

Using Hölder inequality and (3.37) again, we arrive at

$$\int_{\mathbf{R}^3}|\nabla_Q U|^2 e^{-U}\,dQ\int_\Omega|\int_{\mathbf{R}^3}Qge^{-U}\,dQ|^2\,dx \le C|f|_{L^2}^2 = C\int_\Omega\int_{\mathbf{R}^3}g^2e^{-U}\,dQ\,dx$$
$$\le C\int_\Omega\int_{\mathbf{R}^3}|\nabla_Q g|^2e^{-U}\,dQ\,dx, \qquad (3.42)$$

where we used the assumption (2.17) that $\int_{\mathbf{R}^3}|\nabla_Q U|^2e^{-U}\,dQ\le C$. Summing up (3.40) through (3.42) and using the definition of g, we prove (3.36). $\square$

3.2 Higher Order Energy Laws for Mixed Derivatives

In this subsection, we will work on the energy estimates for the mixed derivatives of f. Let $s_2\ge 1, s_1$ be positive integers with $s_1+s_2=s$. We apply $\nabla^{s_1}\nabla_Q^{s_2}$ to the microscopic equation (3.3), and take the L^2 inner product of the resulting equation with $\nabla^{s_1}\nabla_Q^{s_2}f$ to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}^2+|\nabla_Q\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}^2-\frac{1}{2}(\nabla^{s_1}\nabla_Q^{s_2}(\Delta_Q Uf-\frac{1}{2}|\nabla_Q U|^2f),\nabla^{s_1}\nabla_Q^{s_2}f)\\ =&-(\nabla^{s_1}\nabla_Q^{s_2}(u\cdot\nabla f),\nabla^{s_1}\nabla_Q^{s_2}f)-(\nabla^{s_1}\nabla_Q^{s_2}(\nabla uQ\nabla_Q f),\nabla^{s_1}\nabla_Q^{s_2}f) \qquad (3.43)\\ &+(\nabla^{s_1}\nabla_Q^{s_2}(\nabla_Q UQ\sqrt{M}\nabla u),\nabla^{s_1}\nabla_Q^{s_2}f)+\frac{1}{2}(\nabla^{s_1}\nabla_Q^{s_2}(\nabla uQ\nabla_Q Uf),\nabla^{s_1}\nabla_Q^{s_2}f).\end{aligned}$$

Again, we first single out the dissipative terms in (3.43). By Lebnitz's formula and integration by parts, we obtain

$$\begin{aligned}&|\nabla_Q\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}^2-\frac{1}{2}(\nabla^{s_1}(\nabla_Q^{s_2}(\Delta_Q Uf-\frac{|\nabla_Q U|^2}{2}f),\nabla^{s_1}\nabla_Q^{s_2}f)\\ =&|\nabla_Q\nabla^{s_1}\nabla_Q^{s_2}f+\frac{1}{2}\nabla_Q U\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}^2\\ &+\frac{1}{2}\sum_{c_1+c_2=s_2,c_1\ge 1}(\nabla_Q^{c_1}(\Delta_Q U-\frac{|\nabla_Q U|^2}{2})\nabla^{s_1}\nabla_Q^{c_2}f,\nabla^{s_1}\nabla_Q^{s_2}f), \qquad (3.44)\end{aligned}$$

while note by the assumptions (2.18) on U that $|\nabla_Q^{c_1}(\Delta_Q U-\frac{|\nabla_Q U|^2}{2})|\le C(1+|\nabla_Q U|^2)$, we have

$$\begin{aligned}&\sum_{c_1+c_2=s_2,c_1\ge 1}|(\nabla_Q^{c_1}(\Delta_Q U-\frac{|\nabla_Q U|^2}{2})\nabla^{s_1}\nabla_Q^{c_2}f,\nabla^{s_1}\nabla_Q^{s_2}f)| \qquad (3.45)\\ &\le C\sum_{c_2\le s_2-1}\left(|\nabla_Q U\nabla^{s_1}\nabla_Q^{c_2}f|_{L^2}|\nabla_Q U\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}+|\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}|\nabla^{s_1}\nabla_Q^{c_2}f|_{L^2}\right),\end{aligned}$$

which completes the estimate to the dissipation terms in (3.43).

For the first term on the second line of (3.43), due to the incompressibility $\nabla\cdot u=0$, there holds

$$
\begin{aligned}
(\nabla^{s_1}\nabla_Q^{s_2}(u\cdot\nabla f),\nabla^{s_1}\nabla_Q^{s_2}f) &= (\nabla^{s_1}(u\cdot\nabla\nabla_Q^{s_2}f)-u\cdot\nabla\nabla^{s_1}\nabla_Q^{s_2}f,\nabla^{s_1}\nabla_Q^{s_2}f)\\
&= \sum_{d_1+d_2=s_1,d_1\geq 1}(\nabla^{d_1}u\cdot\nabla\nabla^{d_2}\nabla_Q^{s_2}f,\nabla^{s_1}\nabla_Q^{s_2}f). \qquad (3.46)
\end{aligned}
$$

Next, for the terms corresponding to $1\leq d_1\leq 5$ in (3.46), since $s\geq 7$, by Sobolev's Imbedding Theorem, we have

$$
\begin{aligned}
|(\nabla^{d_1}u\cdot\nabla\nabla^{d_2}\nabla_Q^{s_2}f,\nabla^{s_1}\nabla_Q^{s_2}f)| &\leq C|\nabla^{d_1}u|_{L^\infty}|\nabla^{d_2+1}\nabla_Q^{s_2}f|_{L^2}|\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}\\
&\leq C|\nabla u|_{H^s}|\nabla^{d_2+1}\nabla_Q^{s_2}f|_{L^2}|\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}, \qquad (3.47)
\end{aligned}
$$

For the terms corresponding to $d_1>5$ in (3.46), Lemma 3.1 implies that

$$
\begin{aligned}
&|(\nabla^{d_1}u\cdot\nabla\nabla^{d_2}\nabla_Q^{s_2}f,\nabla^{s_1}\nabla_Q^{s_2}f)\\
&\leq C|\nabla^{d_1}u|_{L^2}\sup_{x\in\Omega}(\int_{\mathbf{R}^3}|\nabla^{d_2+1}\nabla_Q^{s_2}f|^2\,dQ)^{1/2}|\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}\\
&\leq C|\nabla u|_{H^s}|\nabla_Q^{s_2}f|_{H^{s_1}}|\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}. \qquad (3.48)
\end{aligned}
$$

Here we note that $d_2+5\leq d_2+d_1\leq s_2$.

Since $d_1\geq 1, d_2+1\leq s_1$, by summing up (3.46)-(3.48), we obtain

$$
|(\nabla^{s_1}\nabla_Q^{s_2}(u\cdot\nabla f),\nabla^{s_1}\nabla_Q^{s_2}f)|\leq C|\nabla u|_{H^s}|\nabla_Q^{s_2}f|_{H^{s_1}}|\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}. \qquad (3.49)
$$

For the other remaining term in the second line of (3.43), one can estimate it as

$$
\begin{aligned}
|(\nabla^{s_1}\nabla_Q^{s_2}(\nabla uQ\nabla_Q f),\nabla^{s_1}\nabla_Q^{s_2}f)| &\leq |(\nabla^{s_1}(\nabla uQ\nabla_Q\nabla_Q^{s_2}f),\nabla^{s_1}\nabla_Q^{s_2}f)|\\
&\quad+|(\nabla^{s_1}(\nabla u\nabla_Q^{s_2}f),\nabla^{s_1}\nabla_Q^{s_2}f)|. \qquad (3.50)
\end{aligned}
$$

By Lebnitz's formula,

$$
(\nabla^{s_1}(\nabla u\nabla_Q^{s_2}f),\nabla^{s_1}\nabla_Q^{s_2}f)=\sum_{d_1+d_2=s_1\leq s-1}(\nabla^{d_1+1}u\nabla^{d_2}\nabla_Q^{s_2}f,\nabla^{s_1}\nabla_Q^{s_2}f), \qquad (3.51)
$$

In the cases that $d_1\leq 4$, one has

$$
\begin{aligned}
|(\nabla^{d_1+1}u\nabla^{d_2}\nabla_Q^{s_2}f,\nabla^{s_1}\nabla_Q^{s_2}f)| &\leq C|\nabla^{d_1+1}u|_{L^\infty}|\nabla^{d_2}\nabla_Q^{s_2}f|_{L^2}|\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}\\
&\leq C|\nabla u|_{H^s}|\nabla^{d_2}\nabla_Q^{s_2}f|_{L^2}|\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2}. \qquad (3.52)
\end{aligned}
$$

While in the cases of $d_1\geq 5$, by Lemma 3.1, we obtain

$$
\begin{aligned}
&|(\nabla^{d_1+1}u\nabla^{d_2}\nabla_Q^{s_2}f,\nabla^{s_1}\nabla_Q^{s_2}f)|\\
&\leq C|\nabla^{d_1+1}u|_{L^2}\sup_{x\in\Omega}(\int_{\mathbf{R}^3}|\nabla^{d_2}\nabla_Q^{s_2}f|^2\,dQ)^{1/2}|\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2} \qquad (3.53)\\
&\leq C|\nabla u|_{H^{s_1}}|\nabla_Q^{s_2}f|_{H^{s_1}}|\nabla^{s_1}\nabla_Q^{s_2}f|_{L^2},
\end{aligned}
$$

where we used the fact that $4+d_2 < d_1+d_2 = s_1$. This finishes the estimate of (3.51).

For the first term on the right hand side of (3.50), note that $s_1 \le s-1$ and $\nabla_Q \cdot (\nabla u Q) = 0$, we obtained, by integration by parts, that

$$
\begin{aligned}
&|(\nabla^{s_1}(\nabla u Q \nabla_Q \nabla_Q^{s_2} f), \nabla^{s_1}\nabla_Q^{s_2} f)| = |(\nabla^{s_1}(\nabla u Q \nabla_Q^{s_2} f), \nabla_Q \nabla^{s_1}\nabla_Q^{s_2} f)| \\
\le &\sum_{d_1+d_2=s_1} |(\nabla_x^{d_1}\nabla u Q \nabla^{d_2}\nabla_Q^{s_2} f, \nabla_Q \nabla^{s_1}\nabla_Q^{s_2} f + \frac{1}{2}\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f)| \\
&+ \sum_{d_1+d_2=s_1} |(\nabla^{d_1}\nabla u Q \nabla^{d_2}\nabla_Q^{s_2} f, \frac{1}{2}\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f)|. \qquad (3.54)
\end{aligned}
$$

In the cases of $d_1 \ge 5$, since $s_1 \le s-1$,

$$
\begin{aligned}
&|(\nabla^{d_1+1} u Q \nabla^{d_2}\nabla_Q^{s_2} f, \nabla_Q \nabla^{s_1}\nabla_Q^{s_2} f + \frac{1}{2}\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f)| \\
\le &|\nabla^{d_1+1} u|_{L^2} \sup_{x\in\Omega} (\int_{\mathbf{R}^3} |Q\nabla^{d_2}\nabla_Q^{s_2} f|^2 \, dQ)^{1/2} |\nabla_Q \nabla^{s_1}\nabla_Q^{s_2} f + \frac{1}{2}\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} \\
\le &|u|_{H^s} |Q\nabla_Q^{s_2} f|_{H^{s_1}} |\nabla_Q \nabla^{s_1}\nabla_Q^{s_2} f + \frac{1}{2}\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}, \qquad (3.55)
\end{aligned}
$$

where we used the fact that $d_2 + 4 \le s_1$.

While in the cases of $d_1 \le 4$,

$$
\begin{aligned}
&|(\nabla^{d_1+1} u Q \nabla^{d_2}\nabla_Q^{s_2} f, \nabla_Q \nabla^{s_1}\nabla_Q^{s_2} f + \frac{1}{2}\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f)| \\
\le &|\nabla^{d_1+1} u|_{L^\infty} |Q\nabla^{d_2}\nabla_Q^{s_2} f|_{L^2} |\nabla_Q \nabla^{s_1}\nabla_Q^{s_2} f + \frac{1}{2}\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} \\
\le &C|u|_{H^s} |\nabla_Q U \nabla^{d_2}\nabla_Q^{s_2} f|_{L^2} |\nabla_Q \nabla^{s_1}\nabla_Q^{s_2} f + \frac{1}{2}\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}. \qquad (3.56)
\end{aligned}
$$

Similar to the proof of (3.55) and (3.56), we find

$$
\begin{aligned}
&\sum_{d_1+d_2=s_1} |(\nabla^{d_1}\nabla u Q \nabla^{d_2}\nabla_Q^{s_2} f \ , \ \frac{1}{2}\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f)| \\
&\qquad\qquad \le C|u|_{H^s} |\nabla_Q U \nabla_Q^{s_2} f|_{H^{s_1}} |\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}. \qquad (3.57)
\end{aligned}
$$

By summing up (3.50) through (3.57) we arrive at

$$
\begin{aligned}
&|(\nabla^{s_1}\nabla_Q^{s_2}(\nabla u Q \nabla_Q f), \nabla^{s_1}\nabla_Q^{s_2} f)| \\
\le &C\Big(|\nabla u|_{H^s} |\nabla_Q^{s_2} f|_{H^{s_1}} |\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} + |u|_{H^s} |\nabla_Q U \nabla_Q^{s_2} f|_{H^{s_1}} (|\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} \\
&+ |\nabla_Q \nabla^{s_1}\nabla_Q^{s_2} f + \frac{1}{2}\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2})\Big). \qquad (3.58)
\end{aligned}
$$

This completes the estimate of (3.50).

Next, let us turn to the first term on the last line of (3.43). Again by Lebnitz's formula, there holds

$$
\begin{aligned}
&|(\nabla^{s_1}\nabla_Q^{s_2}(\nabla u Q\nabla_Q U f), \nabla^{s_1}\nabla_Q^{s_2} f)| \\
\leq & \sum_{d_1+d_2=s_1, c_1+c_2=s_2} |(\nabla_x^{d_1+1} u \nabla_Q^{c_1}(Q\nabla_Q U)\nabla_Q^{c_2}\nabla^{d_2} f, \nabla^{s_1}\nabla_Q^{s_2} f)|. \qquad (3.59)
\end{aligned}
$$

Notice that by (2.18), $|(\nabla_Q^{c_1}(Q\nabla_Q U)| \leq C|Q||\nabla_Q U|$, therefor if $d_1 \leq 4$, one has

$$
\begin{aligned}
&|(\nabla^{d_1+1} u \nabla_Q^{c_1}(Q\nabla_Q U)\nabla_Q^{c_2}\nabla^{d_2} f, \nabla^{s_1}\nabla_Q^{s_2} f)| \\
&\leq |\nabla^{d_1+1} u|_{L^\infty} |Q\nabla_Q^{c_2}\nabla^{d_2} f|_{L^2} |\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}| \\
&\leq C|u|_{H^s} |\nabla_Q U \nabla_Q^{c_2}\nabla^{d_2} f|_{L^2} |\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}. \qquad (3.60)
\end{aligned}
$$

While for the cases of $d_1 \geq 5$, by Lemma 3.1, we have

$$
\begin{aligned}
&|(\nabla^{d_1+1} u \nabla_Q^{c_1}(Q\nabla_Q U)\nabla_Q^{c_2}\nabla^{d_2} f, \nabla^{s_1}\nabla_Q^{s_2} f)| \\
&\leq |\nabla_x^{d_1+1} u|_{L^2} \sup_{x\in\Omega}(\int_{\mathbf{R}^3} |Q\nabla_Q^{c_2}\nabla^{d_2} f|^2\, dQ)^{1/2} |\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} \\
&\leq C|u|_{H^s} |\nabla_Q U \nabla_Q^{c_2} f|_{H^{s_1}} |\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}. \qquad (3.61)
\end{aligned}
$$

Here again, one notices that $d_2 + 4 \leq s_1 \leq s - 1$.

Combining (3.59)-(3.61), we find

$$
\begin{aligned}
&|(\nabla^{s_1}\nabla_Q^{s_2}(\nabla u Q\nabla_Q U f)\ ,\ \nabla^{s_1}\nabla_Q^{s_2} f)| \\
&\qquad \leq C|u|_{H^s} \sum_{c_1\leq s_2} |\nabla_Q U \nabla_Q^{c_2} f|_{H^{s_1}} |\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}. \qquad (3.62)
\end{aligned}
$$

Finally we estimate the last remaining term in (3.43) as follows:

$$
\begin{aligned}
|(\nabla^{s_1}\nabla_Q^{s_2}(\nabla_Q U Q\sqrt{M}\nabla u), \nabla^{s_1}\nabla_Q^{s_2} f)| &= |(\nabla_x^{s_1+1} u \nabla_Q^{s_2}(\nabla_Q U Q\sqrt{M}), \nabla^{s_1}\nabla_Q^{s_2} f)| \\
&\leq C|\nabla u|_{H^{s_1}} |\nabla_Q^{s_2}(\nabla_Q U Q\sqrt{M})|_{L^2} |\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} \\
&\leq C|\nabla u|_{H^{s_1}} |\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}, \qquad (3.63)
\end{aligned}
$$

where we used the assumption (2.17) that $|\nabla_Q^{s_2}(\nabla_Q U\sqrt{M})|_{L^2} \leq C$.

Now let us put together all the mixed derivatives estimates. By combining (3.43)-(3.45), (3.49), (3.62) and (3.63) together, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}^2 + \sigma|\nabla_Q\nabla^{s_1}\nabla_Q^{s_2} f + \frac{1}{2}\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}^2 \qquad (3.64) \\
&\leq C\Bigg(\sum_{c_2\leq s_2-1} [|\nabla_Q U \nabla^{s_1}\nabla_Q^{c_2} f|_{L^2} |\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} + |\nabla^{s_1}\nabla_Q^{c_2} f|_{L^2} |\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}] \\
&\quad + |\nabla_Q^{s_2} f|_{H^{s_1}} |\nabla u|_{H^s} |\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} + |u|_{H^s} |\nabla_Q U \nabla_Q^{s_2} f|_{H^{s_1}} |\nabla_Q \nabla^{s_1}\nabla_Q^{s_2} f \\
&\quad + \frac{1}{2}\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} + |\nabla u|_{H^{s_1}} |\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} \\
&\quad + |u|_{H^s} \sum_{c_1\leq s_2} |\nabla_Q U \nabla_Q^{c_1} f|_{H^{s_1}} |\nabla_Q U \nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}\Bigg).
\end{aligned}
$$

Again to use the dissipative term on (3.64) to control the weighted L^2 estimate of the term $|\nabla_Q U\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2}$, we need a similar Lemma as Lemma 3.2.

Lemma 3.3 *Let $s_2 \geq 1$ be an integer, U satisfies (2.18). Then, there holds*

$$|\nabla_Q U\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} \leq C\sum_{k=0}^{s_2}|\nabla_Q\nabla^{s_1}\nabla_Q^k f+\frac{1}{2}\nabla_Q U\nabla^{s_1}\nabla_Q^k f|_{L^2}, \tag{3.65}$$

$$|\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} \leq C\sum_{k=0}^{s_2-1}|\nabla_Q\nabla^{s_1}\nabla_Q^k f+\frac{1}{2}\nabla_Q U\nabla^{s_1}\nabla_Q^k f|_{L^2}. \tag{3.66}$$

Proof. Again similar to the proof of Lemma 3.2, we set $G = e^{\frac{1}{2}U}\nabla^{s_1}\nabla_Q^{s_2} f$, then by a similar argument as in the proof of (3.39), we find

$$\int_{\mathbf{R}^3}|\nabla_Q U|^2 G^2 e^{-U}\,dQ \leq C\left(\int_{\mathbf{R}^3} G^2 e^{-U}\,dQ + \int_{\mathbf{R}^3}|\nabla_Q G|^2 e^{-U}\,dQ\right). \tag{3.67}$$

However, note by the assumption that $s_2 \geq 1$, then

$$\int_{\mathbf{R}^3} G^2 e^{-U}\,dQ = \int_{\mathbf{R}^3}|\nabla^{s_1}\nabla_Q^{s_2} f|^2\,dQ \tag{3.68}$$
$$\leq C\left(\int_{\mathbf{R}^3}|\nabla_Q\nabla^{s_1}\nabla_Q^{s_2-1} f+\frac{1}{2}\nabla_Q U\nabla^{s_1}\nabla_Q^{s_2-1} f|^2\,dQ + \int_{\mathbf{R}^3}|\nabla_Q U\nabla^{s_1}\nabla_Q^{s_2-1} f|^2\,dQ\right).$$

Summing up (3.67) and (3.68) and integrating the resulting inequality over Ω, we get

$$\begin{aligned}|\nabla_Q U\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} \leq C\Bigg(&|\nabla_Q\nabla^{s_1}\nabla_Q^{s_2} f+\frac{1}{2}\nabla_Q U\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} + |\nabla_Q\nabla^{s_1}\nabla_Q^{s_2-1} f\\ &+\frac{1}{2}\nabla_Q U\nabla^{s_1}\nabla_Q^{s_2-1} f|_{L^2} + |\nabla_Q U\nabla^{s_1}\nabla_Q^{s_2-1} f|_{L^2}\Bigg).\end{aligned} \tag{3.69}$$

A simple induction using the last inequality above, together with (3.35) and (3.69), leads to (3.65).

With (3.65), again note that $s_2 \geq 1$, there holds

$$|\nabla^{s_1}\nabla_Q^{s_2} f|_{L^2} \leq |\nabla_Q\nabla^{s_1}\nabla_Q^{s_2-1} f+\frac{1}{2}\nabla_Q U\nabla^{s_1}\nabla_Q^{s_2-1} f|_{L^2}+\frac{1}{2}|\nabla_Q U\nabla^{s_1}\nabla_Q^{s_2-1} f|_{L^2}, \tag{3.70}$$

which together with (3.65) leads to the conclusion of (3.66). This completes the proof of the Lemma. $\square$

4 Existence

In this section, we will combine all the estimates of the last section to get the global classical solutions of (2.12) and (2.15) provided that the initial velocity is small enough and the initial distribution is close enough to a equilibrium, a Maxwellian distribution. In order to do so, we need to use the basic energy law (2.16). Actually by integrating (2.16) from $[0,t]$, we obtain

$$\begin{aligned} &\int_\Omega \left(\frac{1}{2}|u|^2 + \int_{\mathbf{R}^3} [\psi \ln \psi + U\psi]\; dQ\right)\, dx + \mu \int_0^t |\nabla u|_{L^2}^2\, d\tau \\ \le &\int_\Omega \left(\frac{1}{2}|u_0|^2 + \int_{\mathbf{R}^3} [\psi_0 \ln \psi_0 + U\psi_0]\; dQ\right)\, dx, \end{aligned} \tag{4.1}$$

provided that $\psi > 0$.

While by (3.2) and the fact that $\frac{d^2}{d\tau^2}(\tau \ln \tau) \ge 0$ for $\tau > 0$, if we substitute (3.1) to (4.1), we will obtain:

$$\int_{\mathbf{R}^3} \left[(M + \sqrt{M} f)\ln(M + \sqrt{M} f) + U(M + \sqrt{M} f)\right]\, dQ \ge \int_{\mathbf{R}^3} \sqrt{M} f\, dQ = 0, \tag{4.2}$$

In other words, if $\psi > 0$, (4.1) and (4.2) imply that

$$\frac{1}{2}|u|_{L^2}^2 + \mu \int_0^t |\nabla u|_{L^2}^2\, d\tau \le \frac{1}{2}|u_0|_{L^2}^2 + \int_\Omega \int_{\mathbf{R}^3} [\psi_0 \ln \psi_0 + U\psi_0]\, dQ\, dx \tag{4.3}$$

where $\psi_0 = M + \sqrt{M} f_0$.

Now let us present the main result of this paper:

Theorem 4.1 *Let $s \ge 7$ be an integer, f_0 satisfy (3.2), and $\psi_0 = M + \sqrt{M} f_0 > 0$. Then, there exists a sufficiently small constant ϵ such that*

$$\frac{1}{2}|u_0|_{L^2}^2 + \int_\Omega \int_{\mathbf{R}^3} [\psi_0 \ln \psi_0 + U\psi_0]\, dQ\, dx \le \epsilon^{1+a} \mu\sigma \min(\mu,\sigma), \tag{4.4}$$

$$|u_0|_{H^s}^2 + \|f_0\|_{H^s} + |Qf_0|_{H^4}^2 \le \epsilon \min(\mu,\sigma), \tag{4.5}$$

where $a > 0$ is a small positive constant. Then (2.12) and (2.15) has a unique global classical solution (u,ψ) with $\psi = M + \sqrt{M} f > 0$. and

$$\begin{aligned} \sup_{0\le t<\infty} &\left(|u(t)|_{H^s}^2 + |Qf|_{H^4}^2 + \|f(t)\|_{H^s}^2\right) + \int_0^t \Big(\mu |\nabla u|_{H^s}^2 + \sigma \big[||Q|(\nabla_Q f \\ &+ \frac{1}{2}\nabla_Q U f)|_{H^4}^2 + |\nabla_Q f + \frac{1}{2}\nabla_Q U f|_{H^s}^2\big]\Big)\, d\tau \le C\epsilon \min(\mu,\sigma), \end{aligned} \tag{4.6}$$

Proof. It is rather standard to establish the local existence and uniqueness of smooth solution on $[0,T]$ to (2.12) and (2.15) with the estimate (4.3) and some of the estimates in the previous sections, see also [11, 18].Next we are going to prove that $T = \infty$ provided that ϵ is small enough. Note that $\psi_0 > 0$, then (2.12) and maximal principle imply that $\psi(t,x) > 0$, for $0 \le t \le T$. Therefore, $\ln \psi$ in the first order (basic) energy law (2.16) makes sense, and if we use the substitution $\psi = M + \sqrt{M}f$, the derivation of (4.1)–(4.3), together with (4.4), imply that

$$\frac{1}{2}|u|_{L^2}^2 + \mu\int_0^t |\nabla u|_{L^2}^2\, d\tau \le \epsilon^{1+a}\mu\sigma\min(\mu,\sigma), \tag{4.7}$$

for $0 \le t \le T$.

Let $\eta > 0$ be a fixed small positive constant, we denote

$$\begin{aligned} E_\eta^s &=: \eta|Qf|_{H^4}^2 + \sum_{s_1+s_2=s}\eta^{s_2}|\nabla_Q^{s_2}f|_{H^{s_1}}^2, \\ F_\eta^s &=: \eta||Q|(\nabla_Q f + \frac{1}{2}\nabla_Q U f)|_{H^4}^2 + \sum_{s_1+s_2=s}\eta^{s_2}|\nabla_Q\nabla_Q^{s_2}f + \frac{1}{2}\nabla_Q U\nabla_Q^{s_2}f|_{H^{s_1}}^2. \end{aligned} \tag{4.8}$$

In what follows, η will be such a suitably small positive constant that the following inequality (4.9) will be valid initially at $t = 0$.

Given such a constant η, we let T^* be largest positive constant ($T^* \le T$) such that

$$\sup_{0\le t\le T^*}\left(|u(t)|_{H^s}^2 + E_\eta^s(t)\right) \le 3\epsilon\min(\mu,\sigma). \tag{4.9}$$

By summing up (3.25), (3.33) and (3.64) and using Lemma 3.2, Lemma 3.3, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|u|_{H^s}^2 + E_\eta^s) + \frac{\mu}{2}|\nabla u|_{H^s}^2 + \frac{\sigma}{2}F_\eta^s \\ &\le C\Big\{(|u|_{H^s} + (E_\eta^s)^{\frac{1}{2}})(|\nabla u|_{H^s}^2 + F_\eta^s) \\ &\quad +\eta|\nabla u|_{H^4}|Q\nabla_Q U f|_{H^4} + \sum_{s_1+s_2=s, s_1\le s-1}\eta^{s_2}|\nabla u|_{H^{s_1}}|\nabla_Q^{s_2}f|_{H^{s_1}}\Big\}. \end{aligned} \tag{4.10}$$

Now suppose η is sufficiently small (depending only on σ, μ and the constants in the Lemma 3.1 and the Lemma 3.2), we apply the standard interpolation inequality $|\nabla u|_{H^{s_1}} \le \epsilon|\nabla u|_{L^2} + C_\epsilon|\nabla u|_{H^s}$ for $s_1 \le s-1$, together with (3.36) and (3.66) to find, for $0 \le t \le T^*$, that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|u|_{H^s}^2 + E_\eta^s) + \frac{\mu}{4}|\nabla u|_{H^s}^2 + \frac{\sigma}{4}F_\eta^s \\ &\le \left(C(|u|_{H^s} + (E_\eta^s)^{\frac{1}{2}}) - \frac{\min(\mu,\sigma)}{4}\right)(|\nabla u|_{H^s}^2 + F_\eta^s) + \frac{C|\nabla u|_{L^2}^2}{\sigma}. \end{aligned} \tag{4.11}$$

Note that for ϵ small enough such that $\epsilon < \frac{1}{4C}$, we conclude

$$C(|u|_{H^s} + (E_\eta^s)^{\frac{1}{2}}) - \frac{\min(\mu,\sigma)}{4} \le 0.$$

Therefore by integrating (4.11) over $[0, t]$ with $t \le T^*$, and by (4.9), there holds

$$
\begin{aligned}
(|u|^2_{H^s} + E^s_\eta)(t) + \int_0^t [\frac{\mu}{2}|\nabla u|^2_{H^s} + \frac{\sigma}{2} F^s_\eta]\, d\tau \qquad\qquad (4.12)\\
\le |u_0|_{H^s} + |Qf_0|_{H^4}| + \|f_0\|_{H^s} + + \frac{C}{\sigma}\int_0^t |\nabla u|^2_{L^2}\, d\tau\\
\le (\epsilon + C\epsilon^{1+a})\min(\mu, \sigma) < \frac{5}{2}\epsilon \min(\mu, \sigma), \qquad 0 \le t \le T^*.
\end{aligned}
$$

We used (4.5) and (4.7) in the last step. It is clear that (4.12) contradict with (4.9) unless $T^* = \infty$. Note that for η fixed, E^s_η is equivalent to $|Qf|_{h^4} + \|f\|_{H^s}$ and F^s_η is equivalent to $|Q(\nabla_Q f + \frac{1}{2}\nabla_Q U f)|^2_{H^4} + \|\nabla_Q f + \frac{1}{2}\nabla_Q U f\|^2_{H^s}$. Then from (4.12), we conclude (4.6). This completes the proof of the Theorem 4.1. $\square$

Remark 4.1 *(a). In Theorem 4.1, we assume that $\psi_0 > 0$. Actually we only used this assumption to make $\ln\psi$ in the first order energy to make sense. By an approximation argument, we can replace this assumption by the weaker ones as long as the basic energy law makes sense, such as $\psi_0 \ge 0$.*
(b). Compared with the methods in [15] and [16], here we will have to use the first order energy law. It seems very difficult to control the linear growth term $\sqrt{M}\nabla u Q \cdot \nabla_Q U$ in the microscopic equation of (3.3). This idea was also used in our other work [21] for a similar reason.

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