

# *Global Solutions for Incompressible Viscoelastic Fluids*

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## **Abstract**

We prove the existence of both local and global smooth solutions to the Cauchy problem in the whole space and the periodic problem in the  $n$ -dimensional torus for the incompressible viscoelastic system of Oldroyd-B type in the case of near-equilibrium initial data. The results hold in both two- and three-dimensional spaces. The results and methods presented in this paper are also valid for a wide range of elastic complex fluids, such as magnetohydrodynamics, liquid crystals, and mixture problems.

## **1. Introduction**

Many of the rheological and hydrodynamical properties of complex fluids can be attributed to the competition between the kinetic energies and the internal elastic energies through the special transport properties of their respective internal elastic variables. Moreover, any distortion of microstructures, patterns or configurations in the dynamical flow will involve the deformation tensor  $F$ . In contrast to the classical simple fluids, where the internal energies can be determined solely by the determinant of the deformation tensor  $F$ , the internal energies of complex fluids carry all the information of this tensor [8, 19].

In this paper we consider the following system describing incompressible viscoelastic fluids. The existence results we obtain in this paper, together with the methods, are valid in many related systems, such as those for general polymeric materials [5, 19], magnetohydrodynamics (MHD) [9], liquid crystals [10, 24], and problems of free interface motion in mixtures [38]. The entire coupled hydrodynamical system which we consider here contains a linear momentum equation (force balance law), the incompressibility relation, and a microscopic equation specifying

the special transport of the elastic variable  $F$ :

$$\begin{cases} \nabla \cdot v = 0, \\ v_t + v \cdot \nabla v + \nabla p = \mu \Delta v + \nabla \cdot \left[ \frac{\partial W(F)}{\partial F} F^T \right], \\ F_t + v \cdot \nabla F = \nabla v F. \end{cases} \tag{1}$$

Here  $v(t, x)$  represents the velocity field of materials,  $p(t, x)$  the pressure,  $\mu (>0)$  the viscosity,  $F(t, x)$  the deformation tensor, and  $W(F)$  the elastic energy functional. The third equation is simply the consequence of the chain law. It can also be regarded as the consistence condition of the flow trajectories obtained from the velocity field  $v$  and also of those obtained from the deformation tensor  $F$  [8, 11, 25, 27]. Moreover, on the right-hand side of the momentum equation,  $\frac{\partial W(F)}{\partial F}$  is the Piola–Kirchhoff stress tensor and  $\frac{\partial W(F)}{\partial F} F^T$  is the Cauchy–Green tensor, both in the incompressible case. The latter is the change variable (from Lagrangian to Eulerian coordinates) form of the former one.

Throughout this paper we will adopt the notations of

$$(\nabla v)_{ij} = \frac{\partial v_i}{\partial x_j}, \quad (\nabla v F)_{ij} = (\nabla v)_{ik} F_{kj}, \quad (\nabla \cdot F)_i = \partial_j F_{ij},$$

and summation over repeated indices will always be understood.

The above system is equivalent to the usual Oldroyd-B model for viscoelastic fluids in the case of infinite Weissenberg number [19]. On the other hand, without the viscosity term, it represents exactly the incompressible elasticity in Eulerian coordinates. We refer to [5, 8, 19, 21–23, 26, 27] and their references for the detailed derivation and physical background of the above system.

Due to the elastic nature of our system also being regarded as a first step in understanding the dynamical properties of such systems, the study of the near-equilibrium dynamics of the system is both relevant and very important. For this purpose, we will impose the following initial conditions on system (1):

$$F(0, x) = I + E_0(x), \quad v(0, x) = v_0(x), \quad x \in \Omega, \tag{2}$$

where  $\Omega$  is the physical domain under consideration. We further assume that  $E_0(x)$  and  $v_0(x)$  satisfy the following constraints:

$$\begin{cases} \nabla \cdot v_0 = 0, \\ \det(I + E_0) = 1, \\ \nabla \cdot E_0^T = 0, \\ \nabla_m E_{0ij} - \nabla_j E_{0im} = E_{0lj} \nabla_l E_{0im} - E_{0lm} \nabla_l E_{0ij}. \end{cases} \tag{3}$$

The first three of these expressions are just the consequences of the incompressibility condition [25, 27] and the last one can be understood as the consistency condition for changing variables between the Lagrangian and Eulerian coordinates (see Lemma 3 and Remark 2).

When  $\Omega$  is a bounded domain with smooth boundary, we will choose the following Dirichlet boundary conditions:

$$v(t, x) = 0, \quad E(t, x) = 0, \quad (t, x) \in [0, T) \times \partial\Omega. \tag{4}$$

The global existence of classical solutions for system (1) with small initial data  $E_0$  and  $v_0$ , for the Cauchy problem in the whole space and the periodic problem in the  $n$ -dimensional torus  $\Omega = T^n$ , will be proved in this paper. Our methods in this paper are independent of the space dimensions. We point out that the initial-boundary value problem (1) with (2) and (4) can also be treated using a lengthier procedure, involving a few more technical difficulties than the ones presented in this paper.

There is an extensive history of studies which examine the different phenomena for non-Newtonian fluids, and the results of these investigations have been set forth in the Ericksen–Rivlin models [31, 36], the high-grade fluid models [12, 24, 28], and the LADYZHENSKAYA models [18]. There is an important difference between the system (1) considered here and the models before-mentioned, namely that the system (1) is only a partially dissipative system, which may complicate the process of obtaining the usual existence results for small-data global solutions.

There also exists a vast literature dedicated to the study of compressible nonlinear elasticity [1, 32] and nonlinear wave equations [3, 4, 7, 15, 17, 33]. The powerful techniques employed by the generalized energy methods, which involve the rotation, Lorentz, and scaling invariance, were originally developed by KLAINERMAN for the purpose of studying the solutions to nonlinear wave equations [14]. The methods were later generalized by KLAINERMAN and SIDERIS and expressed in nonrelativistic wave equations and elasticity equations with a smaller number of generators, in the absence of the Lorentz invariance [17, 32]. However, in the case of viscoelasticity, the presence of the viscosity term  $\Delta v$  prevents the system from possessing the scaling invariant properties. Moreover, the incompressibility is in direct violation of the Lorentz invariant properties [34, 35].

In the case of compressible nonlinear elasticity, the special null condition on the energy functional  $W(F)$  (or the nonlinear term in the nonlinear wave equations) has to be imposed to carry out the dispersive estimates for the classical solutions. Due to the presence of the viscosity term  $\Delta v$ , no attempt is made in this paper to establish the dispersive estimates or to understand the nonlinear wave interaction/cancellations using the null conditions in cases such as those given in [32, 34] (although they are under investigation). In fact, we use a kind of standard energy estimate similar to that which is used for the Navier–Stokes equations. The methods in this paper are the higher-order energy estimates, which take advantage of the presence of the dissipative term  $\Delta v$  in the momentum equation and do not take into account the null conditions on the elastic energy function  $W(F)$ . However, due to the absence of the damping mechanism in the transport equation of  $F$ , we have to use a special treatment that reveals the special physical structures of the system. Notice that the usual energy method [13, 20] does not yield the small-data global existence result, since there is no dissipation on the deformation tensor  $F$ . Motivated by the basic energy law (see the next section) and our earlier work in two-dimensional cases [22, 25], we analyze the induced stress term. After the usual expansion around the equilibrium, we notice that  $\nabla \cdot F$  does provide some weak dissipation.

The other key ingredient in this paper is the observation that  $\nabla \times F$  is a high-order term for initial data under our physical considerations. Formally, this is merely

the statement commuted by the Lagrangian partial derivatives commute. Lemma 3 demonstrates the validity of this result in the evolution dynamics of the partial differential equation (PDE) system.

The small-data global existence of the classical solutions for the incompressible viscoelastic system (1) provides us with a better physical understanding of this general system. The proof of the theorems involves all the special couplings between the transport and the induced stress, the incompressibility, and the near-equilibrium expansion. Moreover, the bounds for the initial data (which depend on the viscosity) may also shed some light on the large-Weissenberg-number problem in viscoelasticity.

As for the related studies on the existence of solutions to nonlinear elastic (without viscosity) systems, there are works by SIDERIS [32] and AGEMI [1] on the global existence of classical small solutions to three-dimensional compressible elasticity, under the assumption that the nonlinear terms satisfy the null conditions. The former utilized the generalized energy method together with the additional weighted  $L^2$  estimates, while the latter's proof relies on direct estimations of the fundamental solutions. The global existence for three-dimensional incompressible elasticity was then proved via the incompressible limit method [34] and more recently by a different method [35]. It is worth noticing that SIDERIS and AGEMI used a Eulerian description of the problem, which is equivalent to that in [27, 25]. Global existence for the corresponding two-dimensional problem is still open, and the sharpest related results can be viewed in [3, 4]. For incompressible viscoelastic fluids, LIN et al. [25] proved global existence in the two-dimensional case by introducing an auxiliary vector field to replace the transport variable  $F$ . Their procedure illustrates the intrinsic nature of weak dissipation of the induced stress tensor. LEI & ZHOU [22] obtained the same results via the incompressible limit working directly on the deformation tensor  $F$ . Recently LEI et al. [21] proved global existence for two-dimensional small-strain viscoelasticity without assumptions on the smallness of the rotational part of the initial deformation tensor. Finally, after the completion of this paper, we became aware of a study which investigates similar problems to this paper.

This paper is organized as follows. In Section 2, we review some of the basic concepts in mechanics. Some important properties in both fluid and elastic mechanics will also be presented. Section 3 is devoted to proving local existence. The proof of global existence is completed in Section 4. In Section 5, the incompressible limit is studied. The results obtained by the study may be important for the study of numerical simulations and other engineering applications.

## 2. Basic mechanics of viscoelasticity

In this section, we will explore some of the intrinsic properties of the viscoelastic system presented at the beginning of the paper. These properties reflect the underlying physical origin of the problem and, moreover, are essential to the proof of the global existence result.

Let us recall the definition of the deformation tensor  $F$ . The dynamics of any mechanical problem (under a velocity field), whether in fluids or solids, can be

described by the flow map, a time-dependent family of orientation-preserving diffeomorphisms  $x(t, X)$ ,  $0 \leq t \leq T$ . The material point (labeling)  $X$  in the reference configuration is deformed to the spatial position  $x(t, X)$  at time  $t$ , which is in the observer's coordinate.

The velocity field  $v(t, x)$  determines the flow map, and hence the whole dynamics. However, in order to describe the changes of configurations or patterns that occur during such dynamical processes, we need to define the deformation tensor  $\tilde{F}(t, X)$ :

$$\tilde{F}(t, X) = \frac{\partial x}{\partial X}(t, X). \quad (5)$$

Notice that this quantity is defined in the Lagrangian material coordinate. Obviously it satisfies the following rule [11]:

$$\frac{\partial \tilde{F}(t, X)}{\partial t} = \frac{\partial v(t, x(t, X))}{\partial X}. \quad (6)$$

In the Eulerian coordinate, the corresponding deformation tensor  $F(t, x)$  will be defined as  $F(t, x(t, X)) = \tilde{F}(t, X)$ . Equation (6) will be accordingly transformed into the third equation in system (1) through the chain rule [11, 19, 27]. In the context of the system, it can also be interpreted as the consistency of the flow maps generated by the velocity field  $v$  and deformation field  $F$ .

The difference between fluids and solids lies in the fact that in fluids the internal energy can be determined solely by the determinant part of  $F$  (through density), while in elasticity the energy depends on the whole  $F$ .

The incompressibility can be exactly represented as

$$\det F = 1. \quad (7)$$

The usual incompressible condition  $\nabla \cdot v = 0$ , the first equation in (1), is the direct consequence of this identity.

Since we are interested in small solutions, we define the usual strain tensor by the form

$$E = F - I. \quad (8)$$

The following lemma is well known and appeared in [37]. It illustrates the incompressible consistence of the the system (1).

**Lemma 1.** *Assume that the second equality of (3) is satisfied and  $(v, F)$  is the solution of system (1). Then the following is always true:*

$$\det(I + E) = 1 \quad (9)$$

for all time  $t \geq 0$ .

**Proof.** Using the identity  $\frac{\partial \det F}{\partial F} = (\det F)F^{-T}$ , the first and third equations of (1) give the result

$$\begin{aligned} & (\det(I + E))_t + v \cdot \nabla (\det(I + E)) \\ &= \det(I + E)(I + E)_{ji}^{-1} \nabla_k v_i (I + E)_{kj} \\ &= \det(I + E) \nabla \cdot v = 0. \end{aligned}$$

Thus, the proof of Lemma 1 is completed.

The following lemma played a crucial rule in our earlier work [25, 27]. It provides the third equation in (1) with a div–curl structure of compensate compactness [27], such as that provided by the vorticity equation of the three-dimensional incompressible Euler equations.

**Lemma 2.** *Assume that the third equality of (3) is satisfied, then the solution  $(v, F)$  of the system (1) satisfies the following identities:*

$$\nabla \cdot F^T = 0, \quad \text{and} \quad \nabla \cdot E^T = 0, \tag{10}$$

for all time  $t \geq 0$ .

**Proof.** Following [25, 27], we transpose the third equation of (1) and then apply the divergence operator to the resulting equation to yield

$$\begin{aligned} (\nabla_j F_{ji})_t + v \cdot \nabla (\nabla_j F_{ji}) + \nabla_j v \cdot \nabla F_{ji} \\ = \nabla_j \nabla_k v_j F_{ki} + \nabla_k v_j \nabla_j F_{ki}. \end{aligned}$$

Using the first equation of system (1), we obtain

$$(\nabla_j F_{ji})_t + v \cdot \nabla (\nabla_j F_{ji}) = 0.$$

Thus, the proof Lemma 2 is completed.

**Remark 1.** We can derive the general form of the identity above  $\nabla \cdot F^T = 0$  from the definition of  $\tilde{F}(t, X)$  in (5). However, the proof of the above lemma gives the consistency of the system. The two algebraic identities  $\partial_{X_j} \frac{\partial \det \tilde{F}}{\partial F_{ij}} = 0$  and  $\frac{\partial \det \tilde{F}}{\partial \tilde{F}} = (\det \tilde{F}) \tilde{F}^{-T}$  give the result

$$\partial_{X_j} (\det \tilde{F} \tilde{F}_{ij}^{-T}) = 0. \tag{11}$$

Hence we obtain the following constraint on the deformation tensor  $F$ :

$$\begin{aligned} \nabla_j \left[ \frac{1}{\det F} F_{ij}^T \right] &= F_{jk}^{-T}(t, x) \partial_{X_k} \left[ \frac{1}{\det \tilde{F}} \tilde{F}_{ij}^T(t, X(t, x)) \right] \\ &= \frac{1}{(\det \tilde{F})} \det \tilde{F} \tilde{F}_{jk}^{-T}(t, X(t, x)) \partial_{X_k} \left[ \frac{1}{\det \tilde{F}} \tilde{F}_{ij}^T(t, X(t, x)) \right] \\ &= \frac{1}{\det \tilde{F}} \partial_{X_k} \left[ \tilde{F}_{jk}^{-T}(t, X(t, x)) \tilde{F}_{ij}^T(t, X(t, x)) \right] = 0. \end{aligned}$$

The key ingredient of the later proof in this paper is contained in the following Lemma. It shows that  $\nabla \times E$  is of higher order.

**Lemma 3.** *Assume that the last equality of (3) is satisfied and  $(v, F)$  is the solution of system (1). Then the following identity*

$$\nabla_m E_{ij} - \nabla_j E_{im} = E_{lj} \nabla_l E_{im} - E_{lm} \nabla_l E_{ij} \tag{12}$$

holds for all time  $t \geq 0$ .

**Proof.** To prove the lemma, we will establish the evolution equation for the quantity  $\nabla_m E_{ij} - \nabla_j E_{im} - E_{lj} \nabla_l E_{im} + E_{lm} \nabla_l E_{ij}$ .

First, by the third equation of (1), we can get

$$\begin{aligned} \partial_t \nabla_m E_{ij} + v \cdot \nabla \nabla_m E_{ij} + \nabla_m v \cdot \nabla E_{ij} \\ = \nabla_m \nabla_k v_i E_{kj} + \nabla_k v_i \nabla_m E_{kj} + \nabla_m \nabla_j v_i. \end{aligned} \quad (13)$$

Thus, we have

$$\begin{aligned} \partial_t (\nabla_m E_{ij} - \nabla_j E_{im}) + v \cdot \nabla (\nabla_m E_{ij} - \nabla_j E_{im}) \\ + (\nabla_m v \cdot \nabla E_{ij} - \nabla_j v \cdot \nabla E_{im}) \\ = (\nabla_m \nabla_k v_i E_{kj} - \nabla_j \nabla_k v_i E_{km}) \\ + \nabla_k v_i (\nabla_m E_{kj} - \nabla_j E_{km}). \end{aligned} \quad (14)$$

On the other hand, combining (13) and the third equation of (1), we have

$$\begin{aligned} \partial_t (E_{lm} \nabla_l E_{ij}) + v \cdot \nabla (E_{lm} \nabla_l E_{ij}) \\ = \nabla_l [\nabla_k v_i E_{kj} E_{lm} + \nabla_j v_i E_{lm} + \nabla_m v_l E_{ij}]. \end{aligned}$$

Thus, we get

$$\begin{aligned} \partial_t (E_{lm} \nabla_l E_{ij} - E_{lj} \nabla_l E_{im}) + v \cdot \nabla (E_{lm} \nabla_l E_{ij} - E_{lj} \nabla_l E_{im}) \\ = \nabla_l [\nabla_k v_i (E_{kj} E_{lm} - E_{km} E_{lj}) + (\nabla_j v_i E_{lm} - \nabla_m v_i E_{lj}) \\ + (\nabla_m v_l E_{ij} - \nabla_j v_l E_{im})]. \end{aligned} \quad (15)$$

Combing (14) and (15), we obtain

$$\begin{aligned} \partial_t (\nabla_m E_{ij} - \nabla_j E_{im} + E_{lm} \nabla_l E_{ij} - E_{lj} \nabla_l E_{im}) \\ + v \cdot \nabla (\nabla_m E_{ij} - \nabla_j E_{im} + E_{lm} \nabla_l E_{ij} - E_{lj} \nabla_l E_{im}) \\ = \nabla_l [\nabla_k v_i (E_{kj} E_{lm} - E_{km} E_{lj}) + \nabla_k v_i (\nabla_m E_{kj} - \nabla_j E_{km})] \end{aligned}$$

Using the first equation of (1), we have

$$\begin{aligned} \partial_t (\nabla_m E_{ij} - \nabla_j E_{im} + E_{lm} \nabla_l E_{ij} - E_{lj} \nabla_l E_{im}) \\ + v \cdot \nabla (\nabla_m E_{ij} - \nabla_j E_{im} + E_{lm} \nabla_l E_{ij} - E_{lj} \nabla_l E_{im}) \\ = \nabla_l [\nabla_k v_i (E_{kj} E_{lm} - E_{km} E_{lj}) + v_i (\nabla_m E_{lj} - \nabla_j E_{lm})] \end{aligned}$$

On the other hand, noting (10), this gives

$$\begin{aligned} \partial_t (\nabla_m E_{ij} - \nabla_j E_{im} + E_{lm} \nabla_l E_{ij} - E_{lj} \nabla_l E_{im}) \\ + v \cdot \nabla (\nabla_m E_{ij} - \nabla_j E_{im} + E_{lm} \nabla_l E_{ij} - E_{lj} \nabla_l E_{im}) \\ = \nabla_l [- (v_i E_{kj} \nabla_k E_{lm} - v_i E_{km} \nabla_k E_{lj}) + v_i (\nabla_m E_{lj} - \nabla_j E_{lm})] \\ + \nabla_l [\nabla_k (v_i E_{kj} E_{lm} - v_i E_{km} E_{lj})] \\ = \nabla_l [v_i (\nabla_m E_{lj} - \nabla_j E_{lm} + E_{km} \nabla_k E_{lj} - E_{kj} \nabla_k E_{lm})] \\ + \nabla_l \nabla_k (v_i E_{kj} E_{lm}) - \nabla_l \nabla_k (v_i E_{km} E_{lj}) \\ = \nabla_l [v_i (\nabla_m E_{lj} - \nabla_j E_{lm} + E_{km} \nabla_k E_{lj} - E_{kj} \nabla_k E_{lm})] \end{aligned}$$

Finally, by using (10) and the first equation of (1) once again, we get the evolution of the concerned quantity:

$$\begin{aligned} & \partial_t (\nabla_m E_{ij} - \nabla_j E_{im} + E_{lm} \nabla_l E_{ij} - E_{lj} \nabla_l E_{im}) \\ & \quad + v \cdot \nabla (\nabla_m E_{ij} - \nabla_j E_{im} + E_{lm} \nabla_l E_{ij} - E_{lj} \nabla_l E_{im}) \\ & = \nabla_l v_i (\nabla_m E_{lj} - \nabla_j E_{lm} + E_{km} \nabla_k E_{lj} - E_{kj} \nabla_k E_{lm}) \\ & \quad + v_i (\nabla_l E_{km} \nabla_k E_{lj} - \nabla_l E_{kj} \nabla_k E_{lm}) \\ & = \nabla_l v_i (\nabla_m E_{lj} - \nabla_j E_{lm} + E_{km} \nabla_k E_{lj} - E_{kj} \nabla_k E_{lm}) . \end{aligned}$$

During these calculations, we have used the incompressibility conditions (10) and the first equation of (1) in the second, the third, and the sixth equality. The last equality proves the lemma, since the above quantity will remain zero for all time with zero initial condition.

**Remark 2.** In order to demonstrate the mechanical background of the above lemma, we again go back to the definition of  $\tilde{F}(t, X)$  in (5). Formally, the fact that the Lagrangian derivatives commute yields the fact that  $\partial_{X_k} \tilde{F}_{ij} = \partial_{X_j} \tilde{F}_{ik}$ , which is equivalent to  $\tilde{F}_{lk} \nabla_l F_{ij}(t, x(t, X)) = \tilde{F}_{lj} \nabla_l F_{ik}(t, x(t, X))$ . Thus, one has

$$F_{lk} \nabla_l F_{ij}(t, x) = F_{lj} \nabla_l F_{ik}(t, x),$$

which means that

$$\nabla_k E_{ij} + E_{lk} \nabla_l E_{ij} = \nabla_j E_{ik} + E_{lj} \nabla_l E_{ik}(t, x).$$

This is exactly the result in the above lemma. However, the validity of the statement for any solution of the system (1) is the merit of the above lemma.

Finally, we make some simplifications to system (1).

In addition to their definitions as the elementary symmetric functions of the eigenvalues, the invariants  $\gamma(A)$  of any  $3 \times 3$  matrix  $A$  are conveniently expressed as

$$\gamma_1(A) = \text{tr} A, \quad \gamma_2(A) = \frac{1}{2} \left[ (\text{tr} A)^2 - \text{tr} A^2 \right], \quad \gamma_3(A) = \det A.$$

On the other hand, one can easily get the identity

$$\gamma_3(A + I) = 1 + \gamma_1(A) + \gamma_2(A) + \gamma_3(A).$$

Combining the above identity with (9), one can obtain the incompressible constraint on  $E$  as

$$\text{tr} E = -\det E - \gamma_2(E). \tag{16}$$

By a similar process, the incompressible constraint on  $E$  in two-dimensional case takes

$$\text{tr} E = -\det E. \tag{17}$$

Next, we will consider the isotropic strain energy function  $W(F)$ . We let  $f_1(E)$ ,  $f_2(E)$ , and  $f_3(E)$  represent any generic terms of degree two or higher at the origin.

In the isotropic case,  $W$  depends on  $F$  through the principal invariants of the strain matrix  $FF^T$  [11]. Define the linearized elasticity tensor as

$$A_{lm}^{ij} = \frac{\partial^2 W}{\partial F_{il} \partial F_{jm}}(I). \tag{18}$$

Suppose that the strain energy function  $W(F)$  is isotropic and frame indifferent, the strong Legendre–Hadamard ellipticity condition imposed upon the linearized elasticity tensor (18) takes the form:

$$A_{lm}^{ij} = (\alpha^2 - 2\beta^2)\delta_{il}\delta_{jm} + \beta^2(\delta_{im}\delta_{jl} + \delta_{ij}\delta_{lm}), \quad \text{with } \alpha > \beta > 0, \tag{19}$$

where the positive parameters  $\alpha$  and  $\beta$  depend only on  $W$ . They represent the speeds of propagation of pressure and shear waves, respectively. By (10), (18), and (19), we have

$$\begin{aligned} \nabla_l \left[ \frac{\partial W(F)}{\partial F} F^T \right]_{il} &= \nabla_l \left[ \frac{\partial W(F)}{\partial F} E^T \right]_{il} + \nabla_l \frac{\partial W(F)}{\partial F_{il}} \\ &= \nabla_l \left[ \frac{\partial W(F)}{\partial F} E^T \right]_{il} + \frac{\partial^2 W(I)}{\partial F_{il} \partial F_{jm}} \nabla_l E_{jm} + \nabla_l f_1(E)_{il} \\ &= (\alpha^2 - 2\beta^2) \nabla_i \text{tr} E + \beta^2 \left( \nabla \cdot E^T + \nabla \cdot E \right)_i + \nabla_l f_2(E)_{il} \\ &= \beta^2 (\nabla \cdot E)_i + \nabla_l f_3(E)_{il}, \end{aligned} \tag{20}$$

where we also used the assumptions that the reference configuration is a stress-free state:

$$\frac{\partial W(I)}{\partial F} = 0. \tag{21}$$

Without loss of generality, we assume that the constant  $\beta = 1$ . In particular, in what follows, we only consider the case of Hookean elastic materials:  $\nabla \cdot f_3(E) = \nabla \cdot (EE^T)$ . The system is

$$\begin{cases} \nabla \cdot v = 0, \\ v_t^i + v \cdot \nabla v^i + \nabla_i p = \mu \Delta v^i + E_{jk} \nabla_j E_{ik} + \nabla_j E_{ij}, \\ E_t + v \cdot \nabla E = \nabla v E + \nabla v. \end{cases} \tag{22}$$

All the following proofs and results are also valid for general isotropic elastic energy functions satisfying the strong Legendre–Hadamard ellipticity condition, such as those in (20).

### 3. Local existence

Although the proof of the following local existence theorem is lengthy, the idea is straightforward and has been carried out in the case of two-dimensional Hookean elasticity in [25]. For a self-contained presentation, we will carry out similar proofs for our general cases.

**Theorem 1.** *Let  $k \geq 2$  be a positive integer, and  $v_0, E_0 \in H^k(\Omega)$  which satisfies the incompressible constraint (3). Suppose that the isotropic elastic energy function satisfies the constitutive assumption (19). Then there exists a positive time  $T$ , which depends only on  $\|v_0\|_{H^2}$  and  $\|E_0\|_{H^2}$ , such that the initial value problem or the periodic initial-boundary value problem for (1) [or (22)] has a unique classical solution in the time interval  $[0, T)$  which satisfies*

$$\begin{cases} \partial_t^j \nabla^\alpha v \in L^\infty(0, T; H^{k-2j-|\alpha|}(\Omega)) \cap L^2(0, T; H^{k-2j-|\alpha|+1}(\Omega)), \\ \partial_t^j \nabla^\alpha E \in L^\infty(0, T; H^{k-2j-|\alpha|}(\Omega)). \end{cases} \tag{23}$$

for all  $j, \alpha$  satisfying  $2j + |\alpha| \leq k$ . Moreover, if  $T^* < +\infty$  is the lifespan of the solution, then

$$\int_0^{T^*} \|\nabla v\|_{H^2}^2 dt = +\infty. \tag{24}$$

**Proof.** Using the Galerkin method originally used for the standard Navier–Stokes equation [37] and later modified for a different coupling system [23], we can construct the approximate solutions to the momentum equation of  $v$ , and then substitute this approximate  $v$  into the transport equation to obtain the appropriate solutions of  $E$ . To prove the convergence of the sequence consisting of the approximate solutions we only need a priori estimates for them. For simplicity, we will establish a priori estimates for the smooth solutions of (22). Therefore, let us assume in the rest of this section that  $(v, E)$  is a local smooth solution to system (22) on some time interval  $[0, T)$ .

In this paper,  $\|\cdot\|$  will denote the  $L^2(\Omega)$  norm, where  $\Omega \subseteq \mathbb{R}^n$  will be either an  $n$ -dimensional torus  $T^n$ , or the entire space  $\mathbb{R}^n$  for  $n = 2$  or  $3$ , and  $(\cdot, \cdot)$  the inner product of standard space  $L^2(\Omega)^d$  with  $d \in \{1, 2, 3, 4, 9\}$ .

The original system (1) possesses the following energy law:

$$\frac{d}{dt} \left( \frac{1}{2} \|v\|^2 + \int_{\Omega} (W(F) - W(I)), dx \right) + \mu \|\nabla v\|^2 = 0. \tag{25}$$

Equivalently, for (22), the corresponding energy law will be:

$$\frac{1}{2} \frac{d}{dt} \left( \|v\|^2 + \|E\|^2 \right) + \mu \|\nabla v\|^2 = 0, \tag{26}$$

which follows from the third equation of (22) and the incompressibility.

The following well-known interpolation inequalities are results of the Sobolev embedding theorems and scaling techniques [2, 23]. They will be frequently used in the following higher-order energy estimates.

**Lemma 4.** *Assume  $v \in W^{k,2}(\Omega)$ ,  $k \geq 3$ . The following interpolation inequalities hold.*

1. For  $1 \leq s \leq k$ ,

$$\begin{aligned} \|v\|_{L^4} &\leq C \|v\|^{1-\frac{1}{2s}} \|\nabla^s v\|^{\frac{1}{2s}}, \quad \Omega \subseteq R^2, \\ \|v\|_{L^4} &\leq C \|v\|^{1-\frac{3}{4s}} \|\nabla^s v\|^{\frac{3}{4s}}, \quad \Omega \subseteq R^3, \\ \|\nabla v\|_{L^4} &\leq C \|v\|^{1-\frac{3}{2(s+1)}} \|\nabla^s \nabla v\|^{\frac{3}{2(s+1)}}, \quad \Omega \subseteq R^2, \\ \|\nabla v\|_{L^4} &\leq C \|v\|^{1-\frac{7}{4(s+1)}} \|\nabla^s \nabla v\|^{\frac{7}{4(s+1)}}, \quad \Omega \subseteq R^3, \\ \|\Delta v\|_{L^4} &\leq C \|v\|^{1-\frac{5}{2(s+2)}} \|\nabla^s \Delta v\|^{\frac{5}{2(s+2)}}, \quad \Omega \subseteq R^2, \\ \|\Delta v\|_{L^4} &\leq C \|v\|^{1-\frac{11}{4(s+2)}} \|\nabla^s \Delta v\|^{\frac{11}{4(s+2)}}, \quad \Omega \subseteq R^3, \end{aligned}$$

2. For  $2 \leq s \leq k$ ,

$$\begin{aligned} \|v\|_{L^\infty} &\leq C \|v\|^{1-\frac{1}{s}} \|\nabla^s v\|^{\frac{1}{s}}, \quad \Omega \subseteq R^2, \\ \|v\|_{L^\infty} &\leq C \|v\|^{1-\frac{3}{2s}} \|\nabla^s v\|^{\frac{3}{2s}}, \quad \Omega \subseteq R^3, \\ \|\nabla v\|_{L^\infty} &\leq C \|v\|^{1-\frac{2}{s+1}} \|\nabla^s \nabla v\|^{\frac{2}{s+1}}, \quad \Omega \subseteq R^2, \\ \|\nabla v\|_{L^\infty} &\leq C \|v\|^{1-\frac{5}{2(s+1)}} \|\nabla^s \nabla v\|^{\frac{5}{2(s+1)}}, \quad \Omega \subseteq R^3. \end{aligned}$$

The following two propositions can be found in [2, 16, 23].

**Proposition 1.** *If  $g : R^n \rightarrow R$  is a smooth function with  $g(0) = 0$ , then, for any positive constant  $k$ ,  $g(v) \in L^\infty \cap H^k$  if  $v \in L^\infty \cap H^k$  and*

$$\|g(v)\|_{H^k} \leq C \|v\|_{H^k} \tag{27}$$

for some constant  $C$  depending only on  $g, k$ , and  $\|v\|_{L^\infty}$ .

**Remark 3.** The above proposition is only used in the cases of general elastic energy functions.

**Proposition 2.** *Assume that  $f, g \in H^s(\Omega)$ . Then for any multi-index  $\alpha, |\alpha| \leq s$ , we have*

$$\begin{cases} \|\nabla^\alpha (fg)\| \leq C (\|f\|_{L^\infty} \|\nabla^s g\| + \|g\|_{L^\infty} \|\nabla^s f\|), \\ \|\nabla^\alpha (fg) - f \nabla^\alpha g\| \leq C (\|\nabla f\|_{L^\infty} \|\nabla^{s-1} g\| + \|\nabla g\|_{L^\infty} \|\nabla^{s-1} f\|). \end{cases} \tag{28}$$

for some constant  $C$  depending only on  $n$ .

We divide the proof of the Theorem 1 into two parts.

**Step 1.**  $H^2$  estimate.

Integrating (26) over  $(0, t)$ , one obtains

$$\left( \|v\|^2 + \|E\|^2 \right) + 2\mu \int_0^t \|\nabla v\|^2 dt = \left( \|v_0\|^2 + \|E_0\|^2 \right). \tag{29}$$

By taking the  $L^2$  inner product of the second equation in (22) with  $\Delta v$ , using Lemma 4 and integration by parts, we have

$$\begin{aligned} &\mu \|\Delta v\|^2 \\ &= (v_t, \Delta v) + (v \cdot \nabla v, \Delta v) + (\nabla p, \Delta v) \\ &\quad - (E_{jk} \nabla_j E_{ik}, \Delta v^i) - (\nabla \cdot E, \Delta v) \\ &\leq C \|\Delta v\| (\|v_t\| + \|v\|_{L^\infty} \|\nabla v\| + \|E\|_{L^\infty} \|\nabla E\| + \|\nabla E\|) \\ &\leq C \|\Delta v\| \left( \|v_t\| + \|v\|^{1-\theta(2)} \|\Delta v\|^{\theta(2)} \|\nabla v\| \right. \\ &\quad \left. + \left( \|E\|^{1-\theta(2)} \|\Delta E\|^{\theta(2)} + 1 \right) (\|E\| + \|\Delta E\|) \right) \\ &\leq \frac{1}{2} \mu \|\Delta v\|^2 + g(\|v_t\|, \|\nabla v\|, \|\Delta E\|), \end{aligned}$$

where  $\theta(s)$  ( $0 < \theta(s) < 1$ ) represents a generic function that is determined by Lemma 4, and  $g(\cdot, \cdot, \cdot)$  represents any generic nonnegative and increasing function of its variables. Thus, we have

$$\|\Delta v\|^2 \leq g(\|v_t\|, \|\nabla v\|, \|\Delta E\|). \tag{30}$$

In the meantime, by taking the  $L^2$  inner product of the second equation in (22) with  $v_t$ , using Lemma 4 and integration by parts, we obtain

$$\begin{aligned} &\frac{\mu}{2} \frac{d}{dt} \|\nabla v\|^2 + \|v_t\|^2 \\ &= -(v \cdot \nabla v, v_t) - (\nabla p, v_t) + (E_{jk} \nabla_j E_{ik}, v_t^i) + (\nabla \cdot E, v_t) \\ &\leq \|\nabla v_t\| \left( \|E\|_{L^4}^2 + \|v\|_{L^4}^2 + \|E\| \right) - (E_{ik} \nabla_j E_{jk}, v_t^i) \\ &\leq C \|\nabla v_t\| \left( \|E\|^{1-\theta(2)} \|\Delta E\|^{\theta(2)} + \|v\|^{1-\theta(1)} \|\nabla v\|^{\theta(1)} + \|E\| \right) \\ &\leq \frac{\mu}{8} \|\nabla v_t\|^2 + g(\|v_t\|, \|\nabla v\|, \|\Delta E\|). \end{aligned} \tag{31}$$

In order to obtain the first inequality of the above computation, we used the constraint on  $E$ , which is due to the incompressibility, in Lemma 2.

Next, taking the  $t$  derivative of the second equation in (22), and then taking the  $L^2$  inner product of the resulting equation with  $v_t$ , we can apply Lemma 4 and integration by parts to obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|v_t\|^2 + \mu \|\nabla v_t\|^2 \\ &= -(\partial_t(v \cdot \nabla v), v_t) - (\nabla p_t, v_t) + \left( \partial_t(E_{kj} \partial_k E_{ij}), v_t^i \right) + (\partial_t \partial_j E_{ij}, v_t^i) \\ &= -(v_t \cdot \nabla v, v_t) - \left( \partial_t(E_{kj} E_{ij}), \partial_k v_t^i \right) - (\partial_t E_{ij}, \partial_j v_t^i) \\ &= (v \otimes v_t, \nabla v_t) - \left( \partial_t(EE^T), \nabla v_t \right) - (\partial_t E, \nabla v_t) \\ &\leq \|\nabla v_t\| (\|v_t\| \|v\|_{L^\infty} + \|E_t\| \|E\|_{L^\infty} + \|E_t\|) \end{aligned}$$

$$\begin{aligned}
&\leq \|\nabla v_t\| \left( \|v_t\| \|v\|^{1-\theta(2)} \|\Delta v\|^{\theta(2)} \right. \\
&\quad \left. + \|E_t\| \|E\|^{1-\theta(2)} \|\Delta E\|^{\theta(2)} + \|E_t\| \right) \\
&\leq \frac{\mu}{8} \|\nabla v_t\|^2 + g(\|v_t\|, \|\Delta E\|, \|E_t\|, \|\Delta v\|). \tag{32}
\end{aligned}$$

On the other hand, from the transport equation of (22) we have

$$\begin{aligned}
\|E_t\| &\leq \|E\|_{L^\infty} \|\nabla v\| + \|v\|_{L^\infty} \|\nabla E\| + \|\nabla v\| \\
&\leq C \|E\|^{1-\theta(2)} \|\Delta E\|^{\theta(2)} \|\nabla v\| + \|\nabla v\| \\
&\quad + \|v\|^{1-\theta(2)} \|\Delta v\|^{\theta(2)} (\|E\| + \|\Delta E\|) \\
&\leq g(\|\nabla v\|, \|\Delta E\|, \|\Delta v\|).
\end{aligned}$$

Substituting (30) into the above inequality, one has

$$\|E_t\| \leq g(\|v_t\|, \|\nabla v\|, \|\Delta E\|). \tag{33}$$

Plugging (30) and (33) into (32), one arrives at

$$\frac{1}{2} \frac{d}{dt} \|v_t\|^2 + \frac{7\mu}{8} \|\nabla v_t\|^2 \leq g(\|v_t\|, \|\nabla v\|, \|\Delta E\|). \tag{34}$$

Noting (31) and (34), it is clear that the key now is the estimate of the term  $\|\Delta E\|$ . It follows from the transport equation in (22) that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\Delta E\|^2 \\
&= -(\Delta(v \cdot \nabla E), \Delta E) + (\Delta(\nabla v E), \Delta E) + (\nabla \Delta v, \Delta E) \\
&= -(\Delta(v \cdot \nabla E) - v \cdot \nabla \Delta E, \Delta E) + (\Delta(\nabla v E), \Delta E) + (\nabla \Delta v, \Delta E) \\
&\leq C \|\Delta E\| (\|\Delta E\| \|\nabla v\|_{L^\infty} + \|\Delta v\|_{L^4} \|\nabla E\|_{L^4} + \|\nabla \Delta v\| \|E\|_{L^\infty} + \|\nabla \Delta v\|) \\
&\leq C \|\Delta E\| \left( \|\Delta E\| \|\nabla \Delta v\|^{\theta(2)} \|v\|^{1-\theta(2)} + \|\nabla \Delta v\| \|\Delta E\|^{\theta(2)} \|E\|^{1-\theta(2)} \right. \\
&\quad \left. + \|\nabla \Delta v\|^{\theta(1)} \|v\|^{1-\theta(1)} \|\Delta E\|^{\theta(1)} \|E\|^{1-\theta(1)} + \|\nabla \Delta v\| \right) \\
&\leq g(\|\Delta E\|) \|\nabla \Delta v\|. \tag{35}
\end{aligned}$$

On the other hand, by applying  $\nabla$  to the momentum equation in (22) and then taking the  $L^2$  inner product of the resulting equation with  $\nabla \Delta v$ , we can get

$$\begin{aligned}
&\mu \|\nabla \Delta v\|^2 \\
&= (\nabla v_t, \nabla \Delta v) + (\nabla(v \cdot \nabla v), \nabla \Delta v) + (\nabla \nabla p, \nabla \Delta v) \\
&\quad - \left( \nabla(E_{kj} \partial_k E_{ij}), \nabla \Delta v^i \right) - (\nabla \partial_j E_{ij}, \nabla \Delta v^i) \\
&\leq C \|\nabla \Delta v\| \left( \|\nabla v_t\| + \|\Delta v\| \|v\|_{L^\infty} + \|\nabla v\|_{L^4}^2 \right. \\
&\quad \left. + \|\Delta E\| \|E\|_{L^\infty} + \|\nabla E\|_{L^4}^2 + \|\Delta E\| \right) \\
&\leq C \|\nabla \Delta v\| \left( \|\nabla v_t\| + \|\Delta v\| \|\Delta v\|^{\theta(2)} \|v\|^{1-\theta(2)} + \|v\|^{2-2\theta(1)} \|\Delta v\|^{2\theta(1)} \right. \\
&\quad \left. + \|\Delta E\| \|\Delta E\|^{\theta(2)} \|E\|^{1-\theta(2)} + \|E\|^{2-2\theta(1)} \|\Delta E\|^{2\theta(1)} + \|\Delta E\| \right).
\end{aligned}$$

Using (30) again, this yields

$$\|\nabla \Delta v\| \leq C \|\nabla v_t\| + g(\|v_t\|, \|\nabla v\|, \|\Delta E\|). \tag{36}$$

Inserting (36) into (35), one concludes that

$$\frac{1}{2} \frac{d}{dt} \|\Delta E\|^2 \leq \frac{\mu}{8} \|\nabla v_t\|^2 + g(\|v_t\|, \|\nabla v\|, \|\Delta E\|). \tag{37}$$

Combining (31) and (34) with (37), we arrive at

$$\begin{aligned} & \frac{d}{dt} \left( \|\Delta E\|^2 + \mu \|\nabla v\|^2 + \|v_t\|^2 \right) + \left( \mu \|\nabla v_t\|^2 + \|v_t\|^2 \right) \\ & \leq g(\|v_t\|, \|\nabla v\|, \|\Delta E\|). \end{aligned} \tag{38}$$

It follows from the momentum equation in (22) that

$$\|v_t(0, x)\| \leq C (\|v_0\|_{H^2}, \|E_0\|_{H^2}). \tag{39}$$

Equations (38) and (39) and the Gronwall’s inequality guarantee the fact that there exist positive constants  $T, M_0$ , depending only on  $\|v_0\|_{H^2}, \|E_0\|_{H^2}$  such that

$$\left( \|\Delta E\|^2 + \mu \|\nabla v\|^2 + \|v_t\|^2 \right) + \int_0^T \left( \mu \|\nabla v_t\|^2 + \|v_t\|^2 \right) ds \leq M_0. \tag{40}$$

Returning to (30) and (33), we find that

$$\|\Delta v\| \leq g(M_0), \quad \|E_t\| \leq g(M_0). \tag{41}$$

And recalling (36), we can obtain from (40) that

$$\int_0^T \|\nabla \Delta v\|^2 ds \leq g(M_0). \tag{42}$$

By (29) and (40)–(42), we conclude that there exists a sufficiently large positive constant  $M$  depending only on  $\|v_0\|_{H^2}, \|E_0\|_{H^2}$  such that

$$\left( \|v\|_{H^2}^2 + \|E\|_{H^2}^2 + \|v_t\|^2 + \|E_t\|^2 \right) + \int_0^T \left( \|\nabla v\|_{H^2}^2 + \|v_t\|_{H^1}^2 \right) ds \leq M. \tag{43}$$

We complete the proof of (23) when  $k = 2$ .

To prove (24), we assume that  $T^* < \infty$  is the maximal existence time and

$$\int_0^{T^*} \|\nabla v\|_{H^2}^2 dt < +\infty. \tag{44}$$

Going back to (35), we can use Gronwall’s inequality to get

$$\|\Delta E\| < +\infty, \quad 0 \leq t \leq T^*. \tag{45}$$

On the other hand, by (45) and the transport equation of  $E$  in (22), we have

$$\begin{aligned} \|E_t\|^2 &\leq (\|E\|_{L^\infty}\|\nabla v\| + \|v\|_{L^\infty}\|\nabla E\| + \|\nabla v\|)^2 \\ &\leq K + \|\nabla v\|_{H^2}^2, \quad 0 \leq t \leq T^*. \end{aligned}$$

Thus, by (44), we obtain

$$\int_0^{T^*} \|E_t\|^2 dt < \infty. \tag{46}$$

If we go back to (31), using (29) and (45), we have

$$\begin{aligned} &\frac{\mu}{2} \frac{d}{dt} \|\nabla v\|^2 + \|v_t\|^2 \\ &\leq C \|\nabla v_t\| \left( \|E\|^{1-\theta(2)} \|\Delta E\|^{\theta(2)} \right. \\ &\quad \left. + \|v\|^{1-\theta(1)} \|\nabla v\|^{\theta(1)} + \|E\| \right) \\ &\leq \frac{\mu}{8} \|\nabla v_t\|^2 + C \|\nabla v\|^2 + C. \end{aligned} \tag{47}$$

Similarly, by (29) and (45), (32) will give

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|v_t\|^2 + \mu \|\nabla v_t\|^2 \\ &\leq \|\nabla v_t\| \left( \|v_t\| \|v\|^{1-\theta(2)} \|\Delta v\|^{\theta(2)} \right. \\ &\quad \left. + \|E_t\| \|E\|^{1-\theta(2)} \|\Delta E\|^{\theta(2)} + \|E_t\| \right) \\ &\leq \frac{\mu}{8} \|\nabla v_t\|^2 + C \|v_t\|^2 \left( \|\Delta v\|^2 + 1 \right) + \|E_t\|^2. \end{aligned} \tag{48}$$

Combining (47) and (48), we have

$$\frac{d}{dt} \left( \mu \|\nabla v\|^2 + \|v_t\|^2 \right) + \|v_t\|^2 + \mu \|\nabla v_t\|^2 \leq C \|v_t\|^2 \left( \|\Delta v\|^2 + 1 \right) + \|E_t\|^2.$$

With (44) and (46), we can use Gronwall's inequality to get

$$\left( \|\nabla v\|^2 + \|v_t\|^2 \right) + \int_0^{T^*} \left( \|\nabla v_t\|^2 + \|v_t\|^2 \right) dt < +\infty. \tag{49}$$

Inserting (45) and (49) into (30) and (33), we get

$$\|\Delta v\| < +\infty, \quad \|E_t\| < +\infty, \quad 0 \leq t \leq T^*. \tag{50}$$

Combining (29), (44)–(45), and (49)–(50), we get

$$\left( \|v\|_{H^2}^2 + \|E\|_{H^2}^2 + \|v_t\|^2 + \|E_t\|^2 \right) + \int_0^{T^*} \left( \|\nabla v\|_{H^2}^2 + \|v_t\|_{H^1}^2 \right) ds < +\infty,$$

which contradicts the assumption that  $T^*$  is the maximal existence time, which in turn proves (24) when  $k = 2$ .

**Step 2.** Higher-order energy estimate.

The proof for  $k \geq 2$  is an induction on  $k$ . Assume the theorem is valid for integer  $k$ . In other words, we have

$$\begin{cases} \partial_t^i \nabla^\alpha v \in L^\infty(0, T; H^{k-2i-|\alpha|}(\Omega)) \cap L^2(0, T; H^{k-2i-|\alpha|+1}(\Omega)), \\ \partial_t^i \nabla^\alpha F \in L^\infty(0, T; H^{k-2i-|\alpha|}(\Omega)). \end{cases} \quad (51)$$

for all  $i, \alpha$  satisfying  $2i + |\alpha| \leq k$ ,  $T$  being determined as in step 1. Namely, for all  $i, \alpha$  satisfying  $2i + |\alpha| \leq k$ , we have

$$\left( \|\partial_t^i \nabla^{k-2i} v\|^2 + \|\partial_t^i \nabla^{k-2i} E\|^2 \right) + \int_0^T \|\partial_t^i \nabla^{k+1-2i} v\|^2 dt < +\infty. \quad (52)$$

Here and in what follows the summations are performed over repeated indices  $i$  regardless of their position, as we assumed before. Our goal is to prove that the above results are valid for all  $j, \alpha$  satisfying  $2j + |\alpha| \leq k + 1$ , which are equivalent to:

$$\left( \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^j \nabla^{k+1-2j} E\|^2 \right) + \int_0^T \|\partial_t^j \nabla^{k+2-2j} v\|^2 dt < +\infty, \quad (53)$$

where the summation over  $j$  is from 0 to  $\frac{k}{2}$  if  $k$  is an even number, and from 0 to  $\frac{k+1}{2}$  if  $k$  is an odd number.

First, we assume that  $k$  is an even number and (52) is satisfied. By applying  $\partial_t^j \nabla^{k-2j}$  to the second equation in (22), we have

$$\begin{aligned} & \partial_t^j \nabla^{k-2j} v_t + \partial_t^j \nabla^{k-2j} (v \cdot \nabla v) + \partial_t^j \nabla^{k-2j} \nabla p \\ & = \mu \partial_t^j \nabla^{k-2j} \Delta v + \partial_t^j \nabla^{k-2j} \nabla \cdot (EE^T) + \partial_t^j \nabla^{k-2j} \nabla \cdot E. \end{aligned} \quad (54)$$

By taking the  $L^2$  inner product of (54) with  $\partial_t^{j+1} \nabla^{k-2j} v$ ,  $0 \leq j \leq \frac{k}{2}$  and using integration by parts, we get

$$\begin{aligned} & \frac{\mu}{2} \frac{d}{dt} \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^{j+1} \nabla^{k-2j} v\|^2 \\ & = - \left( \partial_t^j \nabla^{k-2j} (v \cdot \nabla v), \partial_t^{j+1} \nabla^{k-2j} v \right) - \left( \partial_t^j \nabla^{k-2j} \nabla p, \partial_t^{j+1} \nabla^{k-2j} v \right) \\ & \quad + \left( \partial_t^j \nabla^{k-2j} \nabla \cdot (EE^T), \partial_t^{j+1} \nabla^{k-2j} v \right) + \left( \partial_t^j \nabla^{k-2j} \nabla \cdot E, \partial_t^{j+1} \nabla^{k-2j} v \right) \\ & \leq \|\partial_t^{j+1} \nabla^{k-2j} v\| \left( \|\partial_t^j \nabla^{k-2j} (v \cdot \nabla v)\| \right. \\ & \quad \left. + \|\partial_t^j \nabla^{k-2j} \nabla \cdot (EE^T)\| + \|\partial_t^j \nabla^{k-2j} \nabla \cdot E\| \right). \end{aligned} \quad (55)$$

Applying Lemma 4, the induction assumption (52) yields

$$\begin{aligned} & \|\partial_t^j \nabla^{k-2j} (v \cdot \nabla v)\| \\ &= \|\nabla^k (v \cdot \nabla v)\| + \|\partial_t^{\frac{k}{2}} (v \cdot \nabla v)\| + \sum_{0 < j < \frac{k}{2}} \|\partial_t^j \nabla^{k-2j} (v \cdot \nabla v)\| \\ &\leq \|v\|_{L^\infty} \|\nabla^{k+1} v\| + \|v\|_{L^\infty} \|\partial_t^{\frac{k}{2}} \nabla v\| + C \sum_{0 < l \leq \frac{k}{2}} \|\nabla^l v\|_{L^4} \|\nabla^{k+1-l} v\|_{L^4} \\ &\quad + C \sum_{0 < l \leq \frac{k}{2}} \|\partial_t^l v\|_{L^4} \|\partial_t^{\frac{k}{2}-l} \nabla v\|_{L^4} + \|v\|_{L^\infty} \|\partial_t^j \nabla^{k-2j+1} v\| \\ &\quad + C \sum_{0 < j < \frac{k}{2}, (l,n) \neq (j,k-2j)} \|\partial_t^{j-l} \nabla^{k-2j-n} v\|_{L^4} \|\partial_t^l \nabla^n \nabla v\|_{L^4}. \end{aligned}$$

Further computation shows that:

$$\begin{aligned} & \|\partial_t^j \nabla^{k-2j} (v \cdot \nabla v)\| \\ &\leq C \left(1 + \|\partial_t^j \nabla^{k+1-2j} v\|\right) + C \sum_{0 < j < \frac{k}{2}, (l,n) \neq (j,k-2j)} \|\partial_t^{j-l} \nabla^{k-2j-n} v\|^{1-\theta(1)} \\ &\quad \times \|\partial_t^l \nabla^n \nabla v\|^{1-\theta(1)} \cdot \|\partial_t^{j-l} \nabla^{k+1-2j-n} v\|^{\theta(1)} \|\partial_t^l \nabla^{n+1} \nabla v\|^{\theta(1)} \\ &\leq C \left(1 + \|\partial_t^j \nabla^{k+1-2j} v\|\right). \end{aligned} \tag{56}$$

In a similar way, we also have

$$\|\partial_t^j \nabla^{k-2j} \nabla \cdot (EE^T)\| + \|\partial_t^j \nabla^{k-2j} \nabla \cdot E\| \leq C \left(1 + \|\partial_t^j \nabla^{k+1-2j} E\|\right). \tag{57}$$

Putting these estimates (56)–(57) into (55), we obtain

$$\begin{aligned} & \frac{d}{dt} \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^{j+1} \nabla^{k-2j} v\|^2 \\ &\leq C \left(1 + \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^j \nabla^{k+1-2j} E\|\right)^2. \end{aligned} \tag{58}$$

By taking the  $L^2$  inner product of Equation (54) with  $\partial_t^j \nabla^{k-2j} \Delta v$ , for  $0 \leq j \leq \frac{k}{2}$  and using integration by parts, we get

$$\begin{aligned} & \mu \|\partial_t^j \nabla^{k+2-2j} v\|^2 \\ &= \left(\partial_t^{j+1} \nabla^{k-2j} v, \partial_t^j \nabla^{k-2j} \Delta v\right) + \left(\partial_t^j \nabla^{k-2j} (v \cdot \nabla v), \partial_t^j \nabla^{k-2j} \Delta v\right) \\ &\quad - \left(\partial_t^j \nabla^{k-2j} \nabla \cdot (EE^T), \partial_t^j \nabla^{k-2j} \Delta v\right) - \left(\partial_t^j \nabla^{k-2j} \nabla \cdot E, \partial_t^j \nabla^{k-2j} \Delta v\right) \\ &\leq \|\partial_t^j \nabla^{k+2-2j} v\| \left(\|\partial_t^{j+1} \nabla^{k-2j} v\| + \|\partial_t^j \nabla^{k-2j} (v \cdot \nabla v)\| \right. \\ &\quad \left. + \|\partial_t^j \nabla^{k-2j} \nabla \cdot (EE^T)\| + \|\partial_t^j \nabla^{k-2j} \nabla \cdot E\|\right). \end{aligned}$$

Noting (56) and (57), we get

$$\begin{aligned} \|\partial_t^j \nabla^{k+2-2j} v\|^2 &\leq C \left( 1 + \|\partial_t^{j+1} \nabla^{k-2j} v\|^2 \right. \\ &\quad \left. + \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^j \nabla^{k+1-2j} E\|^2 \right). \end{aligned} \tag{59}$$

Applying  $\partial_t^j \nabla^{k+1-2j}$  to the third equation in (22) gives

$$\partial_t^j \nabla^{k+1-2j} E_t + \partial_t^j \nabla^{k+1-2j} (v \cdot \nabla E) = \partial_t^j \nabla^{k+1-2j} (\nabla v E) + \partial_t^j \nabla^{k+1-2j} \nabla v. \tag{60}$$

Now, we take the  $L^2$  inner product of (60) with  $\partial_t^j \nabla^{k+1-2j} E$ ,  $0 \leq j \leq \frac{k}{2}$ , and use integration by parts to obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\partial_t^j \nabla^{k+1-2j} E\|^2 \\ &= \left( \partial_t^j \nabla^{k+1-2j} (\nabla v E), \partial_t^j \nabla^{k+1-2j} E \right) + \left( \partial_t^j \nabla^{k+2-2j} v, \partial_t^j \nabla^{k+1-2j} E \right) \\ &\quad - \left( \partial_t^j \nabla^{k+1-2j} (v \cdot \nabla E), \partial_t^j \nabla^{k+1-2j} E \right) \\ &\leq \|\partial_t^j \nabla^{k+1-2j} E\| \left( \|\partial_t^j \nabla^{k+1-2j} (\nabla v E)\| + \|\partial_t^j \nabla^{k+2-2j} v\| \right) \\ &\quad + \|\partial_t^j \nabla^{k+1-2j} E\| \|\partial_t^j \nabla^{k+1-2j} (v \cdot \nabla E) - v \cdot \nabla \partial_t^j \nabla^{k+1-2j} E\|. \end{aligned} \tag{61}$$

By a similar process as in (56), we can obtain

$$\begin{aligned} &\|\partial_t^j \nabla^{k+1-2j} (\nabla v E)\| \\ &\leq \|\partial_t^{\frac{k}{2}} \nabla (\nabla v E)\| + \|\nabla^{k+1} (\nabla v E)\| + \|\partial_t^j \nabla^{k+2-2j} v\| \|E\|_{L^\infty} \\ &\quad + \|\nabla v\|_{L^\infty} \|\partial_t^j \nabla^{k+1-2j} E\| \\ &\quad + \sum_{0 < j < \frac{k}{2}} \sum_{(l,n) \neq (0,0), (j,k+1-2j)} \|\partial_t^{j-l} \nabla^{k+2-2j-n} v\|_{L^4} \|\partial_t^l \nabla^n E\|_{L^4} \\ &\leq C(1 + \|\nabla \Delta v\|) \left( \|\partial_t^j \nabla^{k+2-2j} v\| + \|\partial_t^j \nabla^{k+1-2j} E\| \right). \end{aligned} \tag{62}$$

On the other hand, we can estimate the last line of (61) as follows

$$\begin{aligned} &\|\partial_t^j \nabla^{k+1-2j} (v \cdot \nabla E) - v \cdot \nabla \partial_t^j \nabla^{k+1-2j} E\| \\ &\leq \|\nabla v\|_{L^\infty} \|\nabla \partial_t^j \nabla^{k-2j} E\| \\ &\quad + C \sum_{(l,n) \neq (j,k-2j), (j,k+1-2j)} \|\partial_t^{j-l} \nabla^{k+1-2j-n} v\|_{L^4} \|\partial_t^l \nabla^{n+1} E\|_{L^4} \\ &\leq C(1 + \|\nabla \Delta v\|) \left( \|\partial_t^j \nabla^{k+2-2j} v\| + \|\partial_t^j \nabla^{k+1-2j} E\| \right). \end{aligned} \tag{63}$$

Combining (59) with (61)–(63), we have

$$\begin{aligned} &\frac{d}{dt} \|\partial_t^j \nabla^{k+1-2j} E\|^2 \\ &\leq C(1 + \|\nabla \Delta v\|) \left( \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^j \nabla^{k+1-2j} E\|^2 \right) \\ &\quad + \frac{1}{2} \|\partial_t^{j+1} \nabla^{k-2j} v\|^2. \end{aligned} \tag{64}$$

Combining this formula (64) with (58), we can conclude that

$$\begin{aligned} & \frac{d}{dt} \left( \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^j \nabla^{k+1-2j} E\|^2 \right) + \|\partial_t^{j+1} \nabla^{k-2j} v\|^2 \\ & \leq C \left( 1 + \|\nabla \Delta v\| \right) \left( \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^j \nabla^{k+1-2j} E\|^2 \right) + C. \end{aligned}$$

Noting that  $\int_0^T |\nabla v|_{L^\infty} dt < \infty$ , we can apply Gronwall’s inequality to get

$$\begin{aligned} & \left( \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^j \nabla^{k+1-2j} E\|^2 \right) \\ & + \int_0^t \|\partial_t^{j+1} \nabla^{k-2j} v\|^2 ds \leq M, \quad 0 \leq t \leq T, \end{aligned} \tag{65}$$

where  $M$  depends only on  $\|v_0\|_{H^{k+1}}$  and  $\|E_0\|_{H^{k+1}}$ . Moreover, by (59), we have

$$\int_0^T \|\partial_t^j \nabla^{k+2-2j} v\|^2 ds \leq M. \tag{66}$$

Together, (65) and (66) imply (53) when  $k$  is an even number.

We now assume that  $k$  is an odd number and  $k \geq 3$ . Applying  $\partial_t^j \nabla^{k+1-2j}$  to the second and third equations of (22), we have

$$\begin{cases} \partial_t^j \nabla^{k+1-2j} v_t + \partial_t^j \nabla^{k+1-2j} (v \cdot \nabla v) + \partial_t^j \nabla^{k+1-2j} \nabla p \\ = \mu \partial_t^j \nabla^{k+1-2j} \Delta v + \partial_t^j \nabla^{k+1-2j} \nabla \cdot (EE^T) + \partial_t^j \nabla^{k+1-2j} \nabla \cdot E, \\ \partial_t^j \nabla^{k+1-2j} E_t + \partial_t^j \nabla^{k+1-2j} (v \cdot \nabla E) = \partial_t^j \nabla^{k+1-2j} (\nabla v E) \\ + \partial_t^j \nabla^{k+1-2j} \nabla v. \end{cases} \tag{67}$$

Now we take the  $L^2$  inner product of the first equation in the system (67) with  $\partial_t^j \nabla^{k+1-2j} v$ , where  $0 \leq j \leq \frac{k+1}{2}$ ; integration by parts yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \mu \|\partial_t^j \nabla^{k+2-2j} v\|^2 \\ & = - \left( \partial_t^j \nabla^{k+1-2j} (v \cdot \nabla v), \partial_t^j \nabla^{k+1-2j} v \right) \\ & + \left( \partial_t^j \nabla^{k+1-2j} \nabla \cdot (EE^T), \partial_t^j \nabla^{k+1-2j} v \right) \\ & + \left( \partial_t^j \nabla^{k+1-2j} \nabla \cdot E, \partial_t^j \nabla^{k+1-2j} v \right) \\ & \leq \|\partial_t^j \nabla^{k+2-2j} v\| \left( \|\partial_t^j \nabla^{k+1-2j} (v \otimes v)\| + \|\partial_t^j \nabla^{k+1-2j} (EE^T)\| \right) \\ & + \|\partial_t^j \nabla^{k+1-2j} E\| \\ & \leq C \|\partial_t^j \nabla^{k+2-2j} v\| \left( 1 + \|\partial_t^j \nabla^{k+1-2j} v\| + \|\partial_t^j \nabla^{k+1-2j} E\| \right), \end{aligned}$$

where we have used Lemma 4 and the induction assumption. In summary, we have

$$\begin{aligned} & \frac{d}{dt} \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \mu \|\partial_t^j \nabla^{k+2-2j} v\|^2 \\ & \leq C \left( 1 + \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^j \nabla^{k+1-2j} E\|^2 \right). \end{aligned} \quad (68)$$

Similarly, we will take the  $L^2$  inner product of the second equation of (67) with  $\partial_t^j \nabla^{k+1-2j} E$ ,  $0 \leq j \leq \frac{k+1}{2}$  and use integration by parts. Similar derivations as in (63) will give

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_t^j \nabla^{k+1-2j} E\|^2 \\ & = - \left( \partial_t^j \nabla^{k+1-2j} (v \cdot \nabla E), \partial_t^j \nabla^{k+1-2j} E \right) + \left( \partial_t^j \nabla^{k+1-2j} (\nabla v E), \partial_t^j \nabla^{k+1-2j} E \right) \\ & \quad + \left( \partial_t^j \nabla^{k+1-2j} \nabla v, \partial_t^j \nabla^{k+1-2j} E \right) \\ & \leq \|\partial_t^j \nabla^{k+1-2j} E\| \left( \|\partial_t^j \nabla^{k+1-2j} (v \cdot \nabla E) - v \cdot \partial_t^j \nabla^{k+1-2j} \nabla E\| \right. \\ & \quad \left. + \|\partial_t^j \nabla^{k+1-2j} (\nabla v E)\| + \|\partial_t^j \nabla^{k+2-2j} v\| \right) \\ & \leq C \|\partial_t^j \nabla^{k+1-2j} E\| \left( 1 + (1 + |\nabla E|_{L^\infty}) \|\partial_t^j \nabla^{k+1-2j} v\| \right. \\ & \quad \left. + (1 + |\nabla v|_{L^\infty}) \|\partial_t^j \nabla^{k+1-2j} E\| + \|\partial_t^j \nabla^{k+2-2j} v\| \right). \end{aligned}$$

Employing the induction assumption, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_t^j \nabla^{k+1-2j} E\|^2 \\ & \leq C \left( 1 + \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^j \nabla^{k+1-2j} E\|^2 \right) + \frac{\mu}{4} \|\partial_t^j \nabla^{k+2-2j} v\|^2. \end{aligned} \quad (69)$$

Combining (68) with (69), we obtain:

$$\begin{aligned} & \frac{d}{dt} \left( \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^j \nabla^{k+1-2j} E\|^2 \right) + \mu \|\partial_t^j \nabla^{k+2-2j} v\|^2 \\ & \leq C \left( 1 + \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^j \nabla^{k+1-2j} E\|^2 \right). \end{aligned}$$

Again, we use Gronwall's inequality to deduce that

$$\begin{aligned} & \left( \|\partial_t^j \nabla^{k+1-2j} v\|^2 + \|\partial_t^j \nabla^{k+1-2j} E\|^2 \right) \\ & \quad + \int_0^t \|\partial_t^j \nabla^{k+2-2j} v\|^2 ds \leq M, \quad 0 \leq t \leq T. \end{aligned} \quad (70)$$

This concludes the proof of (53) when  $k$  is odd.

Putting all these results (65), (66), and (70) together, we have proved (23) and completed the proof of Theorem 1.

### 4. Global existence

We now turn our attention to the proof of the global existence of a classical solution for system (22). A weak dissipation on the deformation  $F$  is found by introducing an auxiliary function  $w$  below. The process of defining such a function reveals the intrinsic dissipative nature of the system.

To avoid complications at the boundary, we only present the periodic case  $\Omega = \mathbf{T}^n$  and the whole-space case  $\Omega = R^n$ . In fact, the case of a smooth-bounded domain can also be treated using a lengthier, but no more difficult, procedure than the proofs presented here.

Unlike previous results in viscoelastic literature [26, 29, 30], the main difficulty lies in the apparent partial dissipation structure of the system (22).

On the other hand, it also lacks the property of scaling invariance. The presence of viscosity on  $v$  is a big obstacle to the use of the combination of Klainerman’s generalized energy estimates and weighted  $L^2$  estimates [15, 17, 32–34].

The main contribution of our work is to reveal the fact that the incompressibility of system (22) provides enough information for the proof of the near-equilibrium global existence of classical solutions.

In the three-dimensional cases, the term  $\nabla \times E$  is in fact a high-order term. We recover the results obtained in [25], where we avoided making use of this fact by introducing the auxiliary vector  $\phi$ , and then show that  $\det \nabla \phi = 1$  is enough to prove the near-equilibrium global existence of classical solutions in the two-dimensional case.

We start the proof by applying  $\Delta$  to the transport equation in (22) and then taking the  $L^2$  inner product of the resulting equation with  $\Delta E$ ,

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|\Delta E\|^2 - (\Delta \nabla v, \Delta E) \\
 &= - (\Delta(v \cdot \nabla E), \Delta E) + (\Delta(\nabla v E), \Delta E) \\
 &\leq C \|\Delta E\| (\|\Delta E\| \|\nabla v\|_{L^\infty} + \|\Delta v\|_{L^4} \|\nabla E\|_{L^4} + \|\nabla \Delta v\| \|E\|_{L^\infty}) \\
 &\leq C \|\Delta E\|^2 (\|\nabla v\| + \|\nabla \Delta v\|) + C \|\Delta E\| \|\nabla \Delta v\| \|E\|_{H^2} \\
 &\quad + C \|\Delta E\| (\|\nabla v\| + \|\nabla \Delta v\|) (\|\Delta E\| + \|E\|) \\
 &\leq C \|E\|_{H^2} \|\Delta E\| (\|\nabla v\| + \|\nabla \Delta v\|) \\
 &\leq C \|E\|_{H^2} \left( \|\Delta E\|^2 + \|\nabla v\|^2 + \|\nabla \Delta v\|^2 \right). \tag{71}
 \end{aligned}$$

Next we apply  $\Delta$  to the momentum equation in (22) and take the  $L^2$  inner of the resulting equation with  $\Delta v$  to deduce that

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|\Delta v\|^2 + \mu \|\nabla \Delta v\|^2 \\
 &= - (\Delta(v \cdot \nabla v), \Delta v) + \left( \Delta \nabla \cdot (EE^T), \Delta v \right) + (\Delta \nabla \cdot E, \Delta v) \\
 &\leq C \|\Delta v\| \|\Delta v\| \|\nabla v\|_{L^\infty} + C \|\Delta E\| \|E\|_{L^\infty} \|\nabla \Delta v\| - (\Delta E, \nabla \Delta v) \\
 &\leq C (\|v\|_{H^2} + \|E\|_{H^2}) \left( \|\nabla v\|^2 + \|\nabla \Delta v\|^2 + \|\Delta E\|^2 \right) \\
 &\quad - (\Delta E, \nabla \Delta v), \tag{72}
 \end{aligned}$$

where in the first inequality we have used Proposition 2.

Combining (71) with (72), we arrive at

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|\Delta v\|^2 + \|\Delta E\|^2 \right) + \mu \|\nabla \Delta v\|^2 \\ & \leq C \left( \|v\|_{H^2} + \|E\|_{H^2} \right) \left( \|\nabla v\|^2 + \|\nabla \Delta v\|^2 + \|\Delta E\|^2 \right). \end{aligned} \quad (73)$$

In order to extract the dissipative nature of the system, we want to combine the linear terms on the right-hand side of the momentum equation in (22). We introduce the auxiliary variable  $w$  as follows:

$$w = \Delta v + \frac{1}{\mu} \nabla \cdot E. \quad (74)$$

The system (22) will give the reformed equation:

$$\begin{aligned} & w_t + \Delta(v \cdot \nabla v) + \frac{1}{\mu} \nabla \cdot (v \cdot \nabla E) + \Delta \nabla p \\ & = \mu \Delta w + \Delta \nabla \cdot (EE^T) + \frac{1}{\mu} \nabla \cdot (\nabla v E) + \frac{1}{\mu} \Delta v. \end{aligned} \quad (75)$$

By taking the  $L^2$  inner product of the resulting equation with  $w$ , we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|w\|^2 + \mu \|\nabla w\|^2 \\ & = - \left( \Delta(v \cdot \nabla v) + \frac{1}{\mu} \nabla \cdot (v \cdot \nabla E), w \right) \\ & \quad - (\Delta \nabla p, w) + \frac{1}{\mu} (\Delta v, w) \\ & \quad + \frac{1}{\mu} (\nabla \cdot (\nabla v E), w) + (\Delta \nabla \cdot (EE^T), w). \end{aligned} \quad (76)$$

Now let us estimate the right-hand side of (76) term by term. First of all, the first term can be estimated as

$$\begin{aligned} & \left| - \left( \Delta(v \cdot \nabla v) + \frac{1}{\mu} \nabla \cdot (v \cdot \nabla E), w \right) \right| \\ & \leq |(v \cdot \nabla w, w)| + |(\Delta(v \cdot \nabla v) - v \cdot \nabla \Delta v, w)| \\ & \quad + \frac{1}{\mu} |(\nabla \cdot (v \cdot \nabla E) - v \cdot \nabla \nabla \cdot E, w)| \\ & \leq |(\Delta(v \otimes v) - v \otimes \Delta v, \nabla w)| \\ & \quad + \frac{1}{\mu} |(\nabla \cdot (v \otimes E) - v \otimes \nabla \cdot E, \nabla w)| \\ & \leq C \left( \|\nabla v\|_{L^4}^2 + \|\Delta v\| \|v\|_{L^\infty} + \frac{1}{\mu} \|\nabla v\| \|E\|_{L^\infty} \right) \|\nabla w\| \\ & \leq C \left( 1 + \frac{1}{\mu} \right) (\|v\|_{H^2} + \|E\|_{H^2}) \left( \|\nabla w\|^2 + \|\nabla v\|^2 + \|\nabla \Delta v\|^2 \right). \end{aligned} \quad (77)$$

Next, we estimate the last term on the right-hand side of (76) as follows:

$$\begin{aligned} & \left| \frac{1}{\mu} (\nabla \cdot (\nabla v E), w) + (\Delta \nabla \cdot (E E^T), w) \right| \\ & \leq C \|\nabla w\| \|E\|_{L^\infty} \left( \|\Delta E\| + \frac{1}{\mu} \|\nabla v\| \right) \\ & \leq C \left( 1 + \frac{1}{\mu} \right) \|E\|_{H^2} \left( \|\nabla w\|^2 + \|\nabla v\|^2 + \|\Delta E\|^2 \right). \end{aligned} \tag{78}$$

Here we have used Proposition 2.

It is rather easy to get

$$\left| \frac{1}{\mu} (\Delta v, w) \right| \leq \frac{\mu}{4} \|\nabla w\|^2 + \frac{C}{\mu^3} \|\nabla v\|^2. \tag{79}$$

Lastly, let us estimate the term  $(\nabla \Delta p, w)$ . Noting that  $\nabla \cdot v = 0$  and (10), by applying the divergence operator to the momentum equation of (22), we get

$$\Delta p = \nabla_j E_{ik} \nabla_i E_{jk} - \nabla_i v_j \nabla_j v_i.$$

By Lemma 4, we have

$$\|\nabla E\|_{L^4}^2 \leq \begin{cases} \|E\|^{\frac{1}{2}} \|\Delta E\|^{\frac{3}{2}} \leq \|E\|_{H^2} \|\Delta E\|, & \text{in } R^2, \\ \|E\|^{\frac{1}{4}} \|\Delta E\|^{\frac{7}{4}} \leq \|E\|_{H^2} \|\Delta E\| & \text{in } R^3, \end{cases} \tag{80}$$

This gives us the following estimates:

$$\begin{aligned} |(\Delta \nabla p, w)| & \leq \|\nabla w\| \left( \|\nabla E\|_{L^4}^2 + \|\nabla v\|_{L^4}^2 \right) \\ & \leq C \left( \|E\|_{H^2} + \|v\|_{H^2} \right) \left( \|\nabla w\|^2 + \|\nabla \Delta v\|^2 + \|\nabla v\|^2 + \|\Delta E\|^2 \right). \end{aligned} \tag{81}$$

Combining all the above laboured but standard estimates (76)–(81) together, we arrive at the following important energy inequality for the auxiliary variable  $w$ :

$$\begin{aligned} & \frac{d}{dt} \|w\|^2 + \mu \|\nabla w\|^2 \\ & \leq \frac{C}{\mu^3} \|\nabla v\|^2 + C \left( 1 + \frac{1}{\mu} \right) \left( \|v\|_{H^2} + \|E\|_{H^2} \right) \\ & \quad \times \left( \|\Delta E\|^2 + \|\nabla v\|^2 + \|\nabla \Delta v\|^2 + \|\nabla w\|^2 \right). \end{aligned} \tag{82}$$

The key here is to estimate the term  $\Delta E$ . Recall the Hodge decomposition

$$\Delta E = \nabla \nabla \cdot E - \nabla \times \nabla \times E.$$

We will now use (12) and (74) to obtain the following estimate:

$$\begin{aligned} \|\Delta E\|^2 & = \|\nabla \nabla \cdot E\|^2 + \|\nabla \times \nabla \times E\|^2 \\ & \leq 2\mu^2 \left( \|\nabla w\|^2 + \|\nabla \Delta v\|^2 \right) + \|\nabla \times \nabla \times E\|^2 \\ & \leq 2\mu^2 \left( \|\nabla w\|^2 + \|\nabla \Delta v\|^2 \right) + C \|E\|_{H^2}^2 \|\Delta E\|^2, \end{aligned} \tag{83}$$

which gives us the bound

$$\|\Delta E\|^2 \leq C\mu^2 \left( \|\nabla w\|^2 + \|\nabla \Delta v\|^2 \right), \tag{84}$$

provided that  $\|E\|_{H^2} \leq \frac{1}{\sqrt{2C}}$ .

With the above result, we are ready to employ the same method as in [25] to prove the global existence results. Combining (73) and (82) with (84), we finally arrive at

$$\begin{aligned} & \frac{d}{dt} \left( \|w\|^2 + \|\Delta E\|^2 + \|\Delta v\|^2 \right) + \mu \left( \|\nabla w\|^2 + \|\nabla \Delta v\|^2 \right) \\ & \leq C \left( \mu^2 + \frac{1}{\mu} \right) \left( \|v\|_{H^2} + \|E\|_{H^2} \right) \left( \|\nabla w\|^2 + \|\nabla v\|^2 + \|\nabla \Delta v\|^2 \right) + \frac{C}{\mu^3} \|\nabla v\|^2. \end{aligned}$$

Thus, if the initial data is sufficiently small, we can find some  $T^* > 0$ , such that

$$\|v\|_{H^2} + \|E\|_{H^2} \leq \frac{\mu^2}{2C(\mu^3 + 1)} \tag{85}$$

for all  $0 \leq t \leq T^*$ . Moreover, in this case,

$$\begin{aligned} & \left( \|w\|^2 + \|\Delta E\|^2 + \|\Delta v\|^2 \right) (t) + \mu \int_0^t \left( \|\nabla w\|^2 + \|\nabla \Delta v\|^2 \right) d\tau \\ & \leq C \left( \mu^2 + \frac{1}{\mu^2} \right) \left( \|v_0\|_{H^2}^2 + \|E_0\|_{H^2}^2 \right) + \frac{C}{\mu^3} \int_0^\infty \|\nabla v\|^2 dt \end{aligned} \tag{86}$$

holds for all  $0 \leq t \leq T^*$ . Noting the original basic energy law (29), we have

$$\left( \|E\|_{H^2}^2 + \|v\|_{H^2}^2 \right) (t) + \mu \int_0^t \|\nabla v\|_{H^2}^2 d\tau \leq C \left( \mu^2 + \frac{1}{\mu^4} \right) \left( \|v_0\|_{H^2}^2 + \|E_0\|_{H^2}^2 \right) \tag{87}$$

holds for all  $0 \leq t \leq T^*$ . Equations (85) and (87) imply that, if

$$\|v_0\|_{H^2}^2 + \|E_0\|_{H^2}^2 < \frac{\mu^8}{8C^3(1 + \mu^6)(1 + \mu^3)^2}, \tag{88}$$

then (85) is still true with  $\leq$  being replaced by  $<$  for all  $0 \leq t \leq T^*$ , which implies that (85) is true for all the latter time with the uniform constant  $C$  independent of  $t$  and  $\mu$ . Moreover, from (87), we have

$$\|E\|_{H^2}^2 + \|v\|_{H^2}^2 + \mu \int_0^\infty \|\nabla v\|_{H^2}^2 dt \leq \frac{\mu^2}{2C(\mu^3 + 1)}.$$

This, together with the local Theorem 1, gives the following global existence of near-equilibrium classical solutions for system (22).

Finally, we state the theorem in slightly more general cases. The proof is exactly the same as in the case of (22).

**Theorem 2.** *Consider the viscoelastic model (1) with the initial data (2) in the whole space  $R^n$  or  $n$ -dimensional torus  $T^n$ , for  $n = 2, 3$ . Suppose that the initial data satisfies the incompressible constraint (3), and the strain energy function satisfies the strong Legendre–Hadamard ellipticity condition (19) as well as the reference configuration stress-free condition (21). Then there exists a unique global classical solution for system (1) that satisfies*

$$\|E\|_{H^2}^2 + \|v\|_{H^2}^2 + \mu \int_0^\infty \|\nabla v\|_{H^2}^2 dt \leq \frac{\mu^2}{2C(\mu^3 + 1)}$$

if the initial data  $v_0, E_0 \in H^k(\Omega)$  and satisfies the condition:

$$\|v_0\|_{H^2}^2 + \|E_0\|_{H^2}^2 < \frac{\mu^8}{M(1 + \mu^{12})},$$

where  $k$  is an integer and  $k \geq 2, M > 8C^3$  is a large enough constant.

### 5. Incompressible limits

In numerical simulations and physical applications, one often views the incompressible system as an approximation of the compressible equations when the Mach number is small enough. Thus, it is of interest to see whether the solution to the incompressible system can be obtained as the incompressible limit of the corresponding compressible system. Moreover, the incompressible limit is also very important in the mathematical understanding of different hydrodynamical systems and has been extensively studied [16, 20, 22, 34].

The corresponding compressible viscoelastic system takes the following form:

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho + \rho \nabla \cdot v = 0, \\ \partial_t v + v \cdot \nabla v + \lambda^2 \frac{p'(\rho)}{\rho} \nabla \rho = \frac{\mu}{\rho} (\Delta v + \nabla(\nabla \cdot v)) + \frac{1}{\rho} \nabla \cdot (\rho F F^T), \\ \partial_t F + v \cdot \nabla F = \nabla u F, \end{cases} \quad (89)$$

where  $p(\rho)$  is a given equation of state independent of the large parameter  $\lambda$  with  $p'(\rho) > 0$  for  $\rho > 0$ , and  $\lambda$  is the reciprocal of the Mach number  $M$ . For simplicity, we only consider the Cauchy problem of system (89). The initial data is

$$\rho^\lambda(0, x) = 1 + \tilde{\rho}_0^\lambda(x), \quad v^\lambda(0, x) = v_0(x) + \tilde{v}_0^\lambda(x), \quad F^\lambda(0, x) = F_0(x) + \tilde{F}_0^\lambda(x), \quad (90)$$

where  $\rho^\lambda(0, x), F^\lambda(0, x)$  satisfy

$$\rho^\lambda(0, x) \det F^\lambda(0, x) = 1,$$

$v_0(x), F_0(x)$  satisfy the incompressible constraints (3), and  $\tilde{\rho}_0^\lambda(x), \tilde{v}_0^\lambda(x), \tilde{F}_0^\lambda(x)$  are assumed to satisfy

$$\|\tilde{\rho}_0^\lambda(x)\|_s \leq \delta_0/\lambda^2, \quad \|\tilde{v}_0^\lambda(x)\|_{s+1} \leq \delta_0/\lambda, \quad \|\tilde{F}_0^\lambda(x)\|_s \leq \delta_0/\lambda. \quad (91)$$

Here  $\delta_0$  is a small positive constant and  $s$  is an integer with  $s \geq 4$ .

For the above system, we can state the following theorem:

**Theorem 3.** *The global classical solution for system (1)–(2) can be viewed as the incompressible limit of system (89)–(90) if (3), (19), (21), and (91) hold and the incompressible initial data satisfies*

$$\|v_0\|_{H^s}^2 + \|E_0\|_{H^s}^2 \leq \varepsilon_0$$

for a sufficiently small constant  $\varepsilon_0$ .

The proof of Theorem 3 relies on the following Lemma 5, namely, the uniform energy estimates with respect to the parameter  $\lambda$ , which was proved in [22] in the two-dimensional case. The methods employed to prove the lemma, as well as the theorem, are very similar to the three-dimensional cases presented here. We will not repeat the process and refer the reader to [22].

**Lemma 5.** *Consider the local solutions of the compressible viscoelastic model (89)–(90) under the constraints (3), (19), (21), and (91). Then the solution  $(\rho^\lambda, v^\lambda, F^\lambda)$  to system (89)–(90) satisfies the following estimates*

$$\begin{cases} E_s(V^\lambda(t)) + \mu \int_0^t \|\nabla v^\lambda\|_s^2 dt \leq C\varepsilon_0, \\ E_{s-1}(\partial_t V^\lambda(t)) + \mu \int_0^t \|\nabla \partial_t v^\lambda\|_{s-1}^2 dt \leq C \exp Ct. \end{cases} \tag{92}$$

for any  $t \in [0, T^\lambda]$  and a universal constant  $C$  independent of  $\lambda$  if the initial data satisfies

$$\|v_0\|_{H^s}^2 + \|E_0\|_{H^s}^2 < \varepsilon_0.$$

Here  $\varepsilon_0$  is a small enough constant and the energy  $E_s(V^\lambda(t))$  is defined as

$$E_s(V^\lambda(t)) = \|\lambda(\rho^\lambda - 1)\|_{H^s}^2 + \|v^\lambda\|_{H^s}^2 + \|E^\lambda\|_{H^s}^2.$$

Moreover  $T_\lambda \rightarrow \infty$ , as  $\lambda \rightarrow +\infty$ .

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