M598B: Homework Assignment 11

Date: Nov. 5, Monday; Due Wed. Nov. 14.

1. Find a solution to

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - au = 0,$$

where a is a constant, with initial condition

$$u(0, x, y) = g(x, y).$$

Hint: Show that $v = e^{-at}u$ satisfies the standard heat equation.

2. Find a solution $u(t, x)$ to

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0, \quad t > 0,\ x > 0, \\ u(0, x) &= g(x), \quad x > 0, (g(0) = 0) \\ u(t, 0) &= 0, \quad t > 0. \end{cases} \tag{1}$$

Hint: Consider the odd extension of the initial data $g(x)$:

$$G(x) = \begin{cases} g(x), & x > 0, \\ -g(-x), & x < 0, \end{cases} \tag{2}$$

and solve the heat equation in the whole line with initial data $G(x)$.

3. Find the fundamental solution ϕ for a dipole source:

$$\Delta\phi(\mathbf{x}) = \nabla\delta(\mathbf{x}) \cdot \mathbf{y} \equiv \lim_{h\to 0} \frac{\delta(\mathbf{x} + h\mathbf{y}) - \delta(\mathbf{x})}{h},$$

where $\mathbf{y}$ is a fixed unit vector. (In electrostatics, a dipole source is like a battery with its two terminals extremely close together in physical space.)

Hint: See Keener (text book), p. 342.

Optional problem. Apply Duhamel's principle to derive a solution to

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) = f(t, x_1, x_2, x_3),$$

with initial condition

$$u(0, x_1, x_2, x_3) = 0.$$

The idea is similar to that of the wave equation.

End.