

1.7. Coordinate transformations.

We deal with coordinate transformations between rectangular coordinate systems, which play an important role in the definition of tensors.

Preliminary remark on tensors. Tensors are physical quantities that exist independent of coordinate systems. Scalar quantities are called zeroth-order tensors (e.g., temperature); vectors are called first-order tensors (e.g., velocity); second-order tensors can all be represented by 3×3 matrices (e.g., the stress tensor), but not all 3×3 matrices are tensors. Tensors of orders greater than 2 cannot be represented by 3×3 matrices. A n -th-order tensor requires 3^n real numbers and is invariant under change of coordinate systems. The requirement of the invariance is natural since physical observables are invariant under change of coordinate systems.

Suppose we have two orthonormal bases:

$$\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, \quad \text{and} \quad \mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3,$$

and two origin O and O' to form two rectangular coordinate systems K and K' . Let a point M in space have the representation

$$\begin{aligned} \mathbf{r} &= x_1 \mathbf{i}_1 + x_2 \mathbf{i}_2 + x_3 \mathbf{i}_3 \\ \mathbf{r}' &= x'_1 \mathbf{i}'_1 + x'_2 \mathbf{i}'_2 + x'_3 \mathbf{i}'_3. \end{aligned} \tag{1}$$

Note the equations:

$$\begin{aligned} \mathbf{r} &= \mathbf{r}' + \mathbf{r}'_0, & \mathbf{r}'_0 &= \vec{OO'} \\ \mathbf{r}' &= \mathbf{r} + \mathbf{r}_0, & \mathbf{r}_0 &= \vec{O'O}, \end{aligned} \tag{2}$$

where the vector \mathbf{r}'_0 is a vector from O to O' and $\mathbf{r}_0 = -\mathbf{r}'_0$. See Figure 1.7.1.

(Figure 1.7.1.)

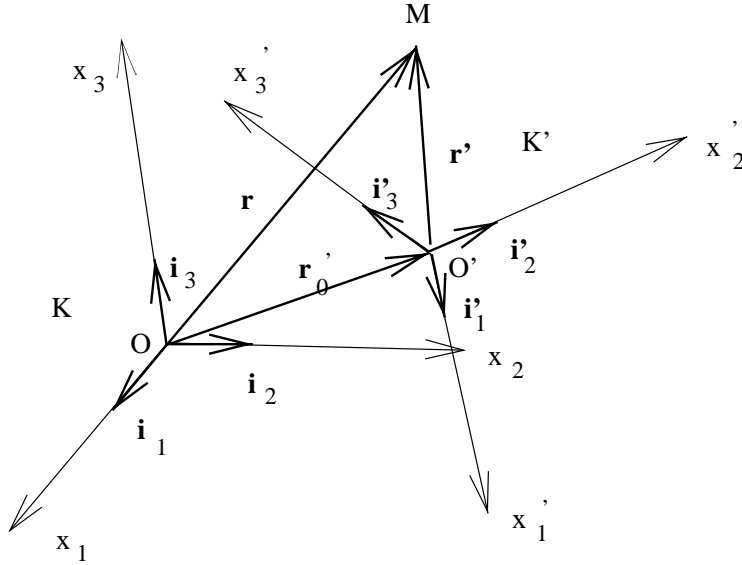


Figure 1.7.1. Coordinate transformation.

Now we use the summation convention: a repeated index is automatically summed:

$$x_k \mathbf{i}_k = \sum_{k=1}^3 x_k \mathbf{i}_k.$$

Thus, the equations in (2) can be written

$$\begin{aligned} x_k \mathbf{i}_k &= x'_k \mathbf{i}'_k + x'_{0k} \mathbf{i}_k \\ x'_k \mathbf{i}'_k &= x_k \mathbf{i}_k + x_{0k} \mathbf{i}'_k \end{aligned} \quad (3)$$

where \$x'_{0k}\$ are the coordinates of \$\mathbf{r}'_0\$ with respect to the old system \$K\$, etc. Take inner product with \$\mathbf{i}_l\$ or \$\mathbf{i}'_l\$ in equations (3) and note the Kronecker delta function

$$\mathbf{i}_k \cdot \mathbf{i}_l = \delta_{kl} = \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases} \quad (4)$$

We find

$$\begin{aligned} x_l &= x'_k (\mathbf{i}'_k \cdot \mathbf{i}_l) + x'_{0l} \\ x'_l &= x_k (\mathbf{i}_k \cdot \mathbf{i}'_l) + x_{0l}. \end{aligned} \quad (5)$$

We introduce new notations

$$\mathbf{i}'_k \cdot \mathbf{i}_l = 1 \cdot 1 \cdot \cos(\mathbf{i}'_k, \mathbf{i}_l) = \alpha_{k'l}. \quad (6)$$

Thus

$$\mathbf{i}_k \cdot \mathbf{i}'_l = \mathbf{i}'_l \cdot \mathbf{i}_k = \alpha_{l'k}. \quad (7)$$

Therefore

$$\begin{aligned} x_l &= \alpha_{k'l} x'_k + x'_{0l} \\ x'_l &= \alpha_{l'k} x_k + x_{0l}. \end{aligned} \quad (8)$$

The first equation of (8) is the transformation from K' to K , while the second equation of (8) is the inverse transformation from K' to K . Note the index summed in the second equation is the second index of α , while the index summed in the first equation of (8) is the first index.

Properties of $\alpha_{l'k}$.

We note that the Kronecker delta in system K' is

$$\delta'_{kl} = \mathbf{i}'_k \cdot \mathbf{i}'_l = \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases} \quad (9)$$

Note the expansions:

$$\mathbf{i}'_k = \alpha_{k'l} \mathbf{i}_l, \quad \text{and} \quad \mathbf{i}_k = \alpha_{l'k} \mathbf{i}'_l,$$

see Figure 1.7.2.

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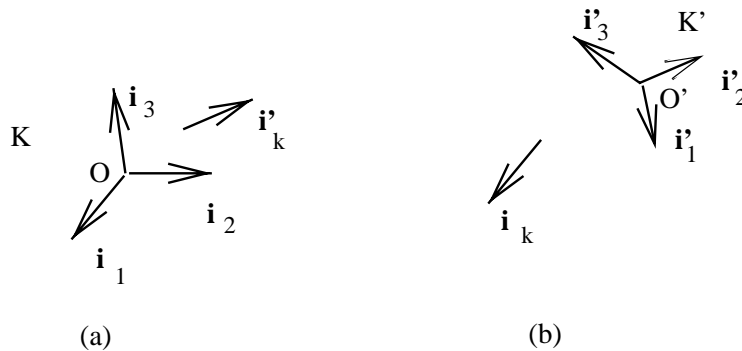


Figure 1.7.2. Calculations of the coefficients.

The $(\alpha_{l'k})$ are often given as a 3×3 matrix

$$(\alpha_{l'k}) = \begin{pmatrix} \alpha_{1'1} & \alpha_{1'2} & \alpha_{1'3} \\ \alpha_{2'1} & \alpha_{2'2} & \alpha_{2'3} \\ \alpha_{3'1} & \alpha_{3'2} & \alpha_{3'3} \end{pmatrix}.$$

Note also

$$\begin{aligned} \delta_{km} &= \mathbf{i}_k \cdot \mathbf{i}_m = \alpha_{l'k} \mathbf{i}'_{l'} \cdot \alpha_{j'm} \mathbf{i}'_j = \alpha_{l'k} \alpha_{l'm} \\ \delta'_{km} &= \mathbf{i}'_k \cdot \mathbf{i}'_m = \alpha_{k'l} \mathbf{i}_l \cdot \alpha_{m'j} \mathbf{i}_j = \alpha_{k'l} \alpha_{m'l}. \end{aligned}$$

Thus we have the property

$$\begin{aligned} \alpha_{l'k} \alpha_{l'm} &= \delta_{km}, \\ \alpha_{k'l} \alpha_{m'l} &= \delta'_{km}. \end{aligned}$$

These properties simply say that the columns of the matrix $(\alpha_{l'k})$ are orthonormal, so are the rows of the matrix. Thus the matrix $(\alpha_{l'k})$ is an orthogonal matrix.