1.4. Theorems of Gauss, Green, and Stokes(Cont'd)

Stokes Theorem Given a surface S bounded by a closed contour L. Suppose that all P, Q, R and their derivatives are continuous on the union of $S \cup L$. Then

$$\begin{aligned}&\iint_S [(\tfrac{\partial R}{\partial x_2} - \tfrac{\partial Q}{\partial x_3}) \cos(\mathbf{n}, x_1) + (\tfrac{\partial P}{\partial x_3} - \tfrac{\partial R}{\partial x_1}) \cos(\mathbf{n}, x_2) + (\tfrac{\partial Q}{\partial x_1} - \tfrac{\partial P}{\partial x_2}) \cos(\mathbf{n}, x_3)] dS \\ &\quad = \oint_L P dx_1 + Q dx_2 + R dx_3,\end{aligned} \tag{1}$$

where $\mathbf{n}$ is the unit normal to S. Here L is traversed in the direction such that S appears to the left of an observer moving along L with the vector $\mathbf{n}$ at points near L pointing from the observer's feet to his/her head.

(Figure 1.4.1. Orientations in Stokes Theorem)

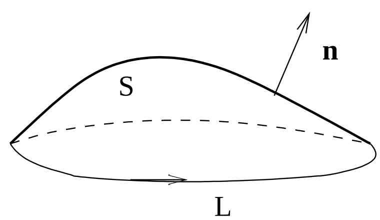

Figure 1.4.1. Orientations in Stokes' Theorem.

Stokes' Theorem in vector form. If we let

$$\mathbf{A} = (P, Q, R) = P\mathbf{i}_1 + Q\mathbf{i}_2 + R\mathbf{i}_3$$

and define

$$\begin{aligned}\text{curl } \mathbf{A} &= (\tfrac{\partial R}{\partial x_2} - \tfrac{\partial Q}{\partial x_3}, \tfrac{\partial P}{\partial x_3} - \tfrac{\partial R}{\partial x_1}, \tfrac{\partial Q}{\partial x_1} - \tfrac{\partial P}{\partial x_2}) \\ &= \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ P & Q & R \end{vmatrix},\end{aligned}$$

then Stokes' Theorem can be written as

$$\int\int_S \text{curl } \mathbf{A} \cdot \mathbf{n} dS = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{r}.$$

We see that the term $\oint_{\partial S} \mathbf{A} \cdot d\mathbf{r}$ is the total circulation of the vector field $\mathbf{A}$ along ∂S. The term $\iint_S \text{curl } \mathbf{A} \cdot \mathbf{n} dS$ is called the total flux of the vector field curl $\mathbf{A}$

through the surface S. In general the total *flux* of a vector $\mathbf{W}$ through a surface S is defined as

$$\int\int_S \mathbf{W} \cdot \mathbf{n} dS.$$

See Figure 1.4.2.

(Figure 1.4.2. Definition of flux)

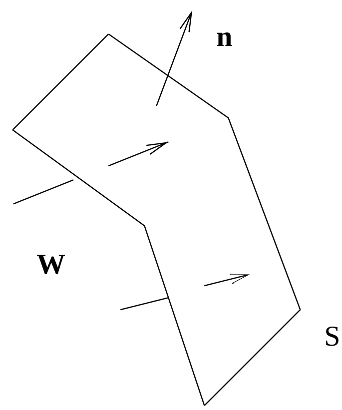

Figure 1.4.2. Flux of a vector field through a surface.

We will come back to the meaning of curl later.

1.4.1. Simply connected domains.

We emphasize that Stokes' Theorem holds only when the vector field $\mathbf{A}$ and its curl are continuous on the union of the surface with its boundary. In general the continuity condition is verified in a domain D that contains S. Sometimes one may make mistakes in the relation of S with D. Let us consider the following question.

Let D be a domain. Suppose a vector $\mathbf{A}$ and all its derivatives are continuous in D. Suppose further that curl $\mathbf{A} = \mathbf{0}$ in D. Can we then use Stokes' Theorem to conclude that $\oint_L \mathbf{A} \cdot d\mathbf{r} = 0$ for any contour L within D?

The answer is yes if D is a solid ball or even a shell (a shell is the region between two concentric balls). But the answer is no if D is a torus. See Figure 1.4.3. To see why, we imagine a contour L that goes along the long circle of the torus. Now it is clear that we can not find a surface S whose boundary is L and lies entirely in the domain D. It may well be the case that the curl of $\mathbf{A}$ is not zero anymore outside of D. In this case, we do not have a surface S to apply Stokes's Theorem on.

(Figure 1.4.3. A torus.)

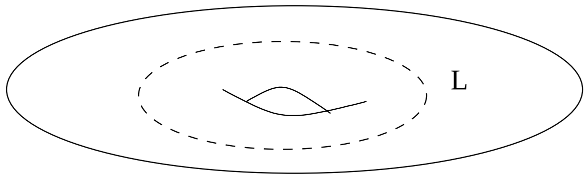

Figure 1.4.3. A torus and a contour that cannot shrink to a point within.

The difference between a torus and a shell or a ball can be characterized as follows. Any closed curve inside a ball can be shrunk within the ball to a point. But not every closed curve inside a torus can be shrunk within the torus to a point.

Definition. A domain is called **simply connected** if any closed curve inside the domain can be shrunk continuously to a point within the domain. A domain is called *multiply connected* if some closed curves cannot be shrunk within the domain to a point.

A torus is multiply connected. A ball is simply connected, so is a shell.

To see how a shell is simply connected, imagine a curve in a domain bounded by two parallel infinite planes. The curve can shrink within the plane to a point. Then imagine bending the plate so that a portion of the plate forms a portion of a shell.

Stokes' Theorem applies to any contour L within a simply connected domain. In particular, circulation of a vector field along any closed curve within a simply connected domain is zero if the vector field and its curl are continuous and the curl vanishes at every point in the domain.

1.5. Scalar fields

Examples of scalar fields are the pressure function $p(\mathbf{r})$ and the temperature function $T(\mathbf{r})$ in a domain D.

A real function of $\mathbf{r}$ in a domain is called a *scalar field*.

1.5.1. Gradient.

Let $h(x_1, x_2)$ denote the height of a mountain for (x_1, x_2) in a planar domain D. The set

$$\{(x_1, x_2) : h(x_1, x_2) = \text{ a height } c\}$$

is called a *level curve.*

The **gradient** of $h(x)$ is defined as

$$\nabla h(x_1, x_2) = (\partial_{x_1} h, \partial_{x_2} h).$$

It is a vector. Its direction gives the direction for fastest change of h. It is normal to the level curve. See Figure 1.5.1.

(Figure 1.5.1.)

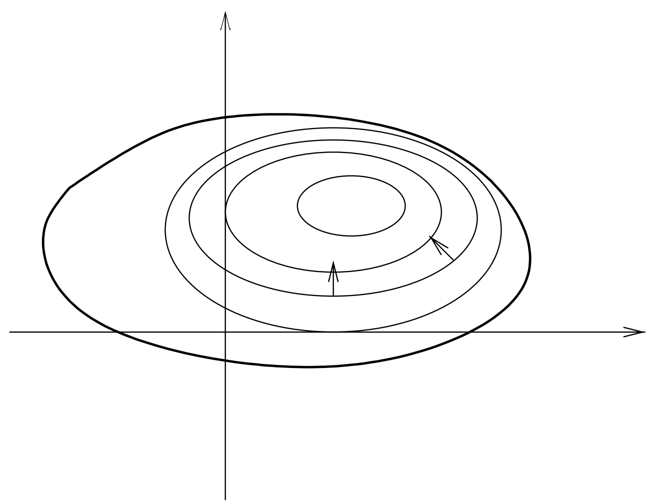

Figure 1.5.1. Level curves and the gradient.

In three dimensions, the gradient of a function $f(x_1, x_2, x_3)$ is defined similarly:

$$\nabla f(x_1, x_2, x_3) = (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f).$$

It is perpendicular to the level surfaces:

$$\{(x_1, x_2, x_3) : f(x_1, x_2, x_3) = c\}$$

where c stands for a real number. The direction of ∇f points to the direction of fastest change of f. See Figure 1.5.2.

(Figure 1.5.2.)

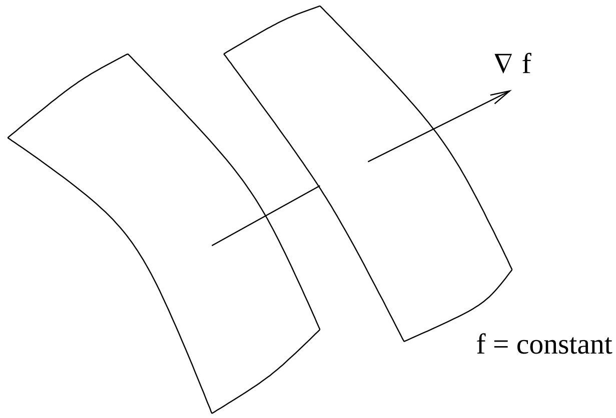

Figure 1.5.2. Level surfaces and the gradient.

End of Lecture 4.