Announcement: The final exam will be cumulative. I will give out a mock exam on the Monday of the exam week on the web.

There will be no class on Dec 7, Friday, it is all because you have been so good.

We will not cover the originally planned topic Homogenization.

We will cover more special functions including Bessel's functions this week, and Green's functions next Monday. Next Wednesday (Dec 5) is a review day.

6.12.3. Example: Heat flow in a nonuniform rod:

$$\text{PDE:} \qquad c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(K_0(x)\frac{\partial u}{\partial x}), \tag{1}$$

$$\text{Boundary condition:} \quad u(t,0) = 0, \tag{2}$$

$$\frac{\partial u}{\partial x}(t,L) = 0, \tag{3}$$

$$\text{Initial condition:} \qquad u(0,x) = g(x). \tag{4}$$

Before jumping to an eigenvalue problem, let us try to use separation of variables:

$$u(t,x) = G(t)\phi(x).$$

Then

$$\frac{G'(t)}{G(t)} = \frac{(K_0(x)\phi')'}{\rho c\phi} = -\lambda.$$

Thus we have the eigenvalue problem

$$\begin{cases} (K_0(x)\phi')' + \lambda\rho c\phi &= 0, \\ \phi(0) = \phi'(L) &= 0. \end{cases}$$

and

$$G(t) = be^{-\lambda t}.$$

By Sturm-Liouville theory, we have

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n \cdots, \tag{5}$$

$$\phi_1, \phi_2, \cdots, \phi_n, \cdots. \tag{6}$$

So general solutions are

$$u(t,x) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t}\phi_n(x).$$

To find b_n, we have the initial condition

$$g(x) = \sum_{n=1}^{\infty} b_n \phi_n(x).$$

Using orthogonality condition with weight ρc, we find

$$\int_0^L g(x)\phi_m(x)\rho c\, dx = b_m \int_0^L \phi_m^2(x)\rho c\, dx, \quad (m = 1, 2, \cdots.)$$

So

$$b_m = \frac{\int_0^L g(x)\phi_m(x)\rho c\, dx}{\int_0^L \phi_m^2(x)\rho c\, dx}. \tag{7}$$

Thus a solution to (1)-(4) is

$$u(t, x) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t}\phi_n(x).$$

where b_n is given in (7), λ_n in (5), and ϕ_n in (6).

6.13. Explicit eigenfunctions: Orthogonal polynomials and special functions.

6.13.1. Legendre polynomials: (p.167, Keener)

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2 - 1)^n]$$

and eigenfunctions of the differential operator

$$\begin{aligned} Lu &= \lambda u, \\ Lu &= -((1 - x^2)u')', \quad -1 < x < 1, \\ \lambda_n &= n(n + 1). \end{aligned}$$

Boundary condition is that u is bounded on $-1 \le x \le 1$. The function $p(x) = 1 - x^2$ vanishes at $|x| = 1$, so it is not regular. (What we covered last time in 6.12 is called a regular Sturm-Liouville eigenvalue problem. But we claim the polynomials are orthogonal and complete nontheless.)

6.13.2. The Schrodinger equation

$$u'' + (E - V(x))u = 0, \quad x \in \mathbb{R}^1.$$

E is eigenvalue and physicist's notation for λ. Let

$$V(x) = x^2 - 1, u = e^{-\frac{x^2}{2}} w.$$

Then

$$w'' - 2xw' + \lambda w = 0,$$

$$w = H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \quad \lambda_n = 2n.$$

The functions $H_n(x)$ are called the **Hermite polynomials**, which are orthogonal polynomials.

See p.167–, Keener's for more special functions.