6.10. Heat equation in a rectangle.

6.10.1. Initial Value Problem

Consider

$$\begin{cases} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \\ u(t,0) &= u(t,L) = 0, \\ u(0,x) &= g(x). \end{cases} \tag{1}$$

We try separation of variables:

$$u(t,x) = \phi(x)G(t). \tag{2}$$

Then

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)}.$$

Thus we set them to be a common constant $-\lambda$:

$$G'(t) = -\lambda k G(t), \tag{3}$$

$$\phi''(x) = -\lambda\phi(x). \tag{4}$$

For ϕ satisfying

$$\phi(0) = \phi(L) = 0, \tag{5}$$

we find the solutions to the **eigenvalue problem**(4)-(5):

$$\phi(x) = c\sin(\frac{n\pi x}{L}), \quad \lambda = (\frac{n\pi}{L})^2, \quad n = 1, 2, \cdots. \tag{6}$$

So we have solutions

$$u(t,x) = \sum_{n=1}^{\infty} C_n e^{-k(\frac{n\pi}{L})^2 t}\sin(\frac{n\pi x}{L}). \tag{7}$$

We need (7) to satisfy the initial condition

$$u(0,x) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L}) = g(x).$$

By Fourier sine series, we only need to take

$$C_n = \frac{2}{L}\int_0^L g(x)\sin(\frac{n\pi x}{L})dx.$$

So a complete solution to (1) is

$$u(t,x) = \sum_{n=1}^{\infty} (\frac{2}{L} \int_0^L g(x) \sin(\frac{n\pi x}{L}) dx) e^{-k(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi x}{L}).$$

6.10.2. Inhomogeneous Problem

Consider

$$\begin{cases} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(t,x), \quad 0 < x < L, \\ u(t,0) &= u(t,L) = 0, \\ u(0,x) &= 0. \end{cases} \tag{8}$$

Let us use the eigenfunctions and propose a solution in the form

$$u(t,x) = \sum_{n=1}^{\infty} C_n(t) \sin(\frac{n\pi x}{L}), \quad C_n(0) = 0.$$

We expand $Q(t,x)$ as

$$Q(t,x) = \sum_{n=1}^{\infty} q_n(t) \sin(\frac{n\pi x}{L}).$$

We note that $Q(t,x)$ may not have zero value at the boundaries, but the expansion is still valid in the $L^2(0,L)$ sense. Then (8) can be projected onto the component $\sin(\frac{n\pi x}{L})$:

$$C_n'(t) = -k(\frac{n\pi}{L})^2 C_n + q_n(t).$$

We find

$$C_n(t) = e^{-k(\frac{n\pi}{L})^2 t} \int_0^t e^{k(\frac{n\pi}{L})^2 s} q_n(s) ds.$$

Thus a solution to (8) is

$$u(t,x) = \sum_{n=1}^{\infty} e^{-k(\frac{n\pi}{L})^2 t} (\int_0^t e^{k(\frac{n\pi}{L})^2 s} q_n(s) ds) \sin(\frac{n\pi x}{L}),$$

where

$$q_n(t) = \frac{2}{L} \int_0^L Q(t,x) \sin(\frac{n\pi x}{L}) dx. \quad (n = 1, 2, \cdots)$$

6.10.3. Boundary Value Problem

Consider

$$\begin{cases} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \\ u(0,x) &= 0, \\ u(t,0) &= a(t), \\ u(t,L) &= b(t). \end{cases} \tag{9}$$

We use the variable

$$V = u - [a(t) + \frac{x}{L}(b(t) - a(t))].$$

Then

$$\begin{cases} \frac{\partial V}{\partial t} &= k\frac{\partial^2 V}{\partial x^2} - [a'(t) + \frac{x}{L}(b'(t) - a'(t))], \quad 0 < x < L, \\ v(0,x) &= -[a(0) + \frac{x}{L}(b(0) - a(0))], \\ v(t,0) &= 0, \\ v(t,L) &= 0. \end{cases}$$

This can be solved by the previous two steps.

We shall solve the heat equation in a two-dimensional rectangular domain next time.