6.3. (Continued)

For the inhomogeneous problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) = f(t, x_1, x_2, x_3), \tag{1}$$

$$u(0, \vec{x}) = 0, \tag{2}$$

$$\frac{\partial u}{\partial t}(0, \vec{x}) = 0, \tag{3}$$

a solution is given by Duhamel's principle (Fritz John, PDE, p.135)

$$u(t, \vec{x}) = \frac{1}{4\pi c^2} \int_0^t \frac{ds}{t-s} \int\int_{|y-x|=c(t-s)} f(s, \vec{y}) dS_y. \tag{4}$$

Duhamel's principle: Given a time $t > 0$. Replace force $f(s, \vec{x}), s \in [0, t]$, by acquired velocity at time

$$0 = s_1 < s_2 < s_3 < \cdots < s_n < s_{n+1} = t,$$

and consider $w_i(s, \vec{x})$:

$$\frac{\partial^2 w_i}{\partial s^2} - c^2 \left(\frac{\partial^2 w_i}{\partial x_1^2} + \frac{\partial^2 w_i}{\partial x_2^2} + \frac{\partial^2 w_i}{\partial x_3^2}\right) = 0, \quad s > s_i, \tag{5}$$

$$w_i(s_i, \vec{x}) = 0, \tag{6}$$

$$\frac{\partial w_i}{\partial s}(s_i, \vec{x}) = f(s_i, \vec{x})(s_{i+1} - s_i). \tag{7}$$

The solution $w_i(s, \vec{x})$, which we assume is zero for $s < s_i$, is the part of the displacement $u(t, \vec{x})$ that is resulted from a pulse force $f(s, \vec{x})$ during the time interval $[s_i, s_{i+1}]$, which is equivalent to a velocity $f(s_i, \vec{x})(s_{i+1} - s_i)$. The final total displacement $u(t, \vec{x})$ is by superposition

$$u(t, \vec{x}) = \sum_{i=1}^{n} w_i(t, \vec{x}). \tag{8}$$

Let $n \to \infty$ and all $s_{i+1} - s_i \to 0$, the approximation becomes exact. We can solve (5)-(7) just as before (Poisson formula):

$$w_i(s, \vec{x}) = \frac{1}{4\pi c^2 (s - s_i)} \int\int_{|\vec{y}-\vec{x}|=c(s-s_i)} (s_{i+1} - s_i) f(s_i, \vec{y}) dS_y, \quad s > s_i.$$

Details are in John, PDE, p.135.

Applications: Maxwell's equations of electromagnetism $(\vec{E}, \vec{B})$ in vacuum are

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2(\frac{\partial^2 \vec{E}}{\partial x_1^2} + \frac{\partial^2 \vec{E}}{\partial x_2^2} + \frac{\partial^2 \vec{E}}{\partial x_3^2}) = 0,$$

$$\frac{\partial^2 \vec{B}}{\partial t^2} - c^2(\frac{\partial^2 \vec{B}}{\partial x_1^2} + \frac{\partial^2 \vec{B}}{\partial x_2^2} + \frac{\partial^2 \vec{B}}{\partial x_3^2}) = 0,$$

where $c = (\frac{T}{\rho})^{\frac{1}{2}}$ is the speed of light in vacuum. We see that the speed of light is lower in air since the density ρ is higher.

6.4. Hadamard's method of descent.

In $I\!R^2$, the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}) = 0,$$

$$u(0, x_1, x_2) = g(x_1, x_2),$$

$$\frac{\partial u}{\partial t}(0, x_1, x_2) = h(x_1, x_2),$$

can be regarded as a problem in $I\!R^3$ where $u(t, x_1, x_2, x_3)$ is independent of the third dimension x_3. In this way, we find that the spherical integrals on the sphere

$$|\vec{y} - \vec{x}| = ((y_1 - x_1)^2 + (y_2 - x_2)^2 + y_3^2)^{\frac{1}{2}} = ct$$

can be changed into top and bottom integrals over the disk

$$(y_1 - x_1)^2 + (y_2 - x_2)^2 < (ct)^2.$$

Thus

$$u(t, x_1, x_2) = \frac{1}{2\pi c}\int\int_{r<ct} \frac{h(y_1, y_2)}{(c^2t^2 - r^2)^{\frac{1}{2}}} dy_1 dy_2 + \frac{\partial}{\partial t}[\frac{1}{2\pi c}\int\int_{r<ct} \frac{g(y_1, y_2)}{(c^2t^2 - r^2)^{\frac{1}{2}}} dy_1 dy_2],$$

where $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

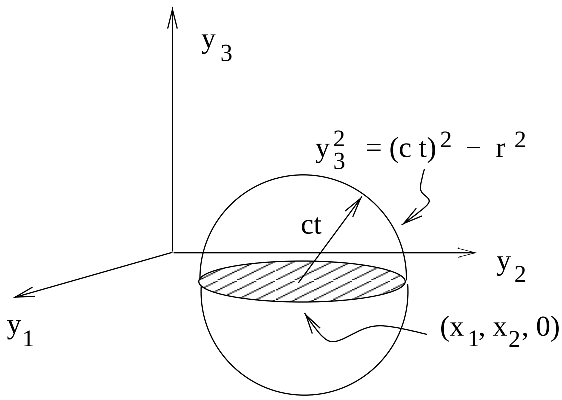

Figure 6.4.1. Integrals on a sphere becomes integrals on a disk.

Nonlinear wave equations

Large amplitude:

$$u_{tt} - c^2 \left(\frac{u_x}{(1-u_x^2)^{\frac{1}{2}}}\right)_x = 0,$$

In liquid crystals:

$$u_{tt} - c(u)(c(u)u_x)_x = 0,$$

where $c(u) = (\alpha \cos^2 u + \beta \sin^2 u)^{\frac{1}{2}}, \quad \alpha > 0, \quad \beta > 0$, are physical elastic constants. These nonlinear equations do not have solution formulas.