

## Chapter VI. Partial Differential Equations

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### 6.1 Transport equation, method of characteristics

We consider the simplest partial differential equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, \quad t > 0, \quad x \in \mathbb{R}^1, \quad (1)$$

where  $a$  is a constant. The general solution formula is

$$u(t, x) = g(x - at) \quad (2)$$

where  $g(\cdot)$  is an arbitrary (smooth) function. Let  $t = 0$  in (2), we see that

$$u(0, x) = g(x), \quad (3)$$

thus  $g(\cdot)$  is the initial condition for  $u$  and equation (1). One can let  $g$  be a Gaussian:  $g(x) = e^{-x^2}$  and plot the solution at times  $t = 1, 2, 3, \dots, 10$  for  $a = -2, -1, 0, 1, 2$ . We can conclude that the graph of  $u(t, x)$  is simply the graph of  $g(x)$  shifted by the amount  $at$  in the  $x$  direction.

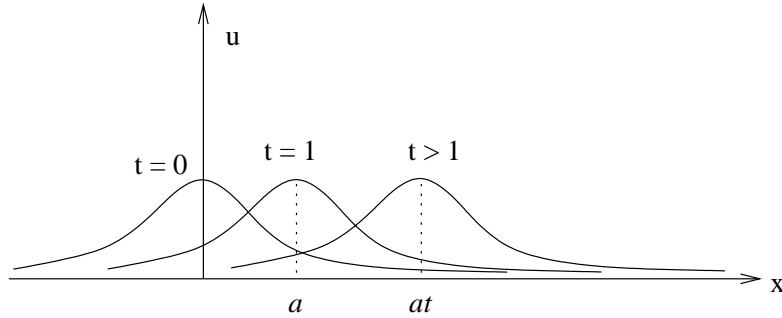


Figure 6.1. Transport feature ( shown for positive velocity  $a$ ).

We consider now the *transport equation* in  $n$ -dimension

$$\frac{\partial u}{\partial t} + a_1 \frac{\partial u}{\partial x_1} + a_2 \frac{\partial u}{\partial x_2} + \dots + a_n \frac{\partial u}{\partial x_n} = 0, \quad t > 0, \quad \vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n \quad (4)$$

with initial condition

$$u(0, \vec{x}) = g(\vec{x}). \quad (5)$$

It can be readily verified that

$$u(t, \vec{x}) = g(\vec{x} - \vec{a}t), \quad (6)$$

Equation (4) is called a *passive transport equation*. We can add a source term to it and consider

$$\begin{cases} \frac{\partial u}{\partial t} + \vec{a} \cdot \nabla u = f(t, \vec{x}), \\ u(0, \vec{x}) = g(\vec{x}). \end{cases} \quad (7)$$

Let us consider the straight lines

$$\frac{d\vec{x}}{dt} = \vec{a}, \quad (8)$$

i.e.,

$$\vec{x} = \vec{x}(t) \equiv \vec{x}_0 + \vec{a}t, \quad (9)$$

which cover the whole space  $\mathbb{R}^n \times \mathbb{R}$ , when  $\vec{x}_0$  and  $t$  vary freely. These lines are called characteristic lines of equation (7). See Figure 6.2. Let us fix a  $\vec{x}_0$  and consider the function  $u(t, \vec{x}(t))$ . We find

$$\frac{d}{dt}u(t, \vec{x}(t)) = \frac{\partial u}{\partial t} + \nabla u \cdot \frac{d}{dt}\vec{x}(t) = \frac{\partial u}{\partial t} + a \cdot \nabla u = f(t, \vec{x}(t)). \quad (10)$$

Thus we can integrate (10) to find

$$u(t, \vec{x}(t)) = u(0, \vec{x}(0)) + \int_0^t f(s, \vec{x}(s)) ds = g(\vec{x}_0) + \int_0^t f(s, \vec{x}_0 + \vec{a}s) ds. \quad (11)$$

Looking at the characteristic lines the other way around, we can first fix a point  $(t, \vec{x}) \in \mathbb{R}^1 \times \mathbb{R}^n$ , and determine an  $\vec{x}_0$  at  $t = 0$  from (9), and then (11) reads as

$$u(t, \vec{x}) = g(\vec{x} - \vec{a}t) + \int_0^t f(s, \vec{x} - \vec{a}(t-s)) ds.$$

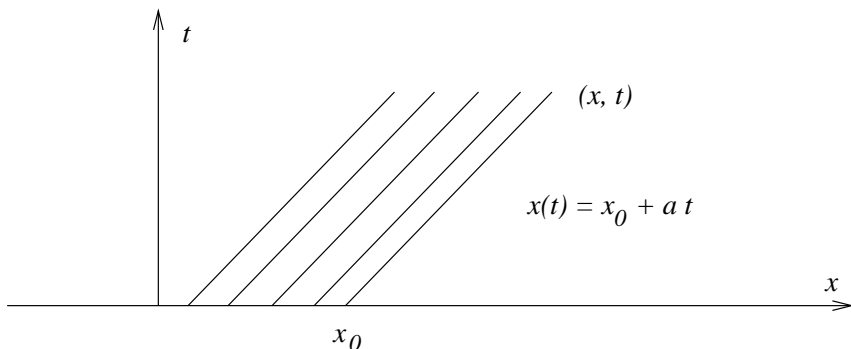


Figure 6.2. Characteristic lines .

### Motivation of the equation:

Convection or transport is an important part in many partial differential equations, such as neutron transport, Boltzmann equation, fluid dynamics, etc.

The method used in (8)-(11) is called the *method of characteristics*. This method can be used to solve equation (7) when  $\vec{a}$  is a function of  $(t, \vec{x})$ , or even when  $\vec{a}$  is a function of  $u$ , making (7) a nonlinear first-order equation.