Chapter VI. Partial Differential Equations

Tentative contents

6.1 Transport equation, method of characteristics

We consider the simplest partial differential equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad t > 0, \quad x \in I\!R^1, \tag{1}$$

where a is a constant. The general solution formula is

$$u(t, x) = g(x - at) \tag{2}$$

where $g(\cdot)$ is an arbitrary (smooth) function. Let $t = 0$ in (2), we see that

$$u(0, x) = g(x), \tag{3}$$

thus $g(\cdot)$ is the initial condition for u and equation (1). One can let g be a Gaussian: $g(x) = e^{-x^2}$ and plot the solution at times $t = 1, 2, 3, \cdots, 10$ for $a = -2, -1, 0, 1, 2$. We can conclude that the graph of $u(t, x)$ is simply the graph of $g(x)$ shifted by the amount at in the x direction.

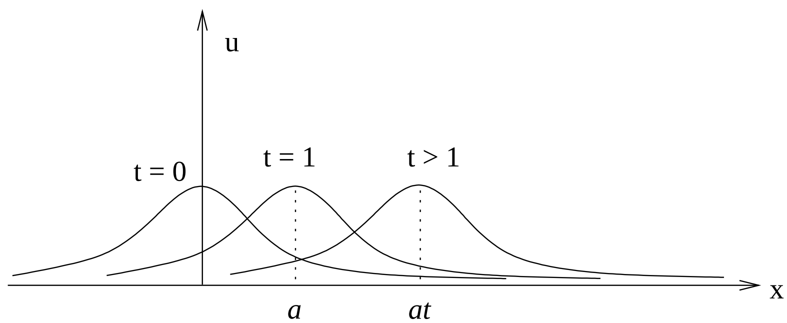

Figure 6.1. Transport feature (shown for positive velocity *a*).

We consider now the *transport equation* in n-dimension

$$\frac{\partial u}{\partial t} + a_1 \frac{\partial u}{\partial x_1} + a_2 \frac{\partial u}{\partial x_2} + \cdots + a_n \frac{\partial u}{\partial x_n} = 0, \quad t > 0, \quad \vec{x} = (x_1, \cdots, x_n) \in I\!R^n \tag{4}$$

with initial condition

$$u(0, \vec{x}) = g(\vec{x}). \tag{5}$$

It can be readily verified that

$$u(t, \vec{x}) = g(\vec{x} - \vec{a}t), \tag{6}$$

Equation (4) is called a *passive transport equation.* We can add a source term to it and consider

$$\begin{cases} \frac{\partial u}{\partial t} + \vec{a} \cdot \nabla u = f(t, \vec{x}), \\ \quad u(0, \vec{x}) = g(\vec{x}). \end{cases} \tag{7}$$

Let us consider the straight lines

$$\frac{d\vec{x}}{dt} = \vec{a}, \tag{8}$$

i.e.,

$$\vec{x} = \vec{x}(t) \equiv \vec{x}_0 + \vec{a}t, \tag{9}$$

which cover the whole space $I\!R^n \times I\!R$, when $\vec{x}_0$ and t vary freely. These lines are called characteristic lines of equation (7). See Figure 6.2. Let us fix a $\vec{x}_0$ and consider the function $u(t, \vec{x}(t))$. We find

$$\frac{d}{dt}u(t, \vec{x}(t)) = \frac{\partial u}{\partial t} + \nabla u \cdot \frac{d}{dt}\vec{x}(t) = \frac{\partial u}{\partial t} + a \cdot \nabla u = f(t, \vec{x}(t)). \tag{10}$$

Thus we can integrate (10) to find

$$u(t, \vec{x}(t)) = u(0, \vec{x}(0)) + \int_0^t f(s, \vec{x}(s))ds = g(\vec{x}_0) + \int_0^t f(s, \vec{x}_0 + \vec{a}s)\, ds. \tag{11}$$

Looking at the characteristic lines the other way around, we can first fix a point $(t, \vec{x}) \in I\!R^1 \times I\!R^n$, and determine an $\vec{x}_0$ at $t = 0$ from (9), and then (11) reads as

$$u(t, \vec{x}) = g(\vec{x} - \vec{a}t) + \int_0^t f(s, \vec{x} - \vec{a}(t - s))ds.$$

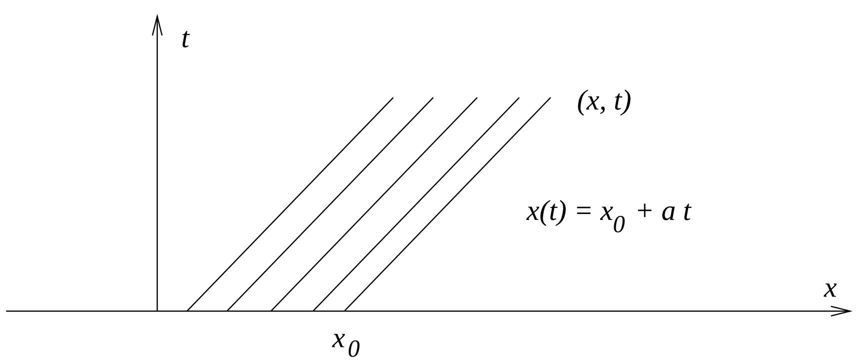

Figure 6.2. Characteristic lines .

Motivation of the equation:

Convection or transport is an important part in many partial different equations, such as neutron transport, Boltzmann equation, fluid dynamics, etc.

The method used in (8)-(11) is called the *method of characteristics.* This method can be used to solve equation (7) when $\vec{a}$ is a function of $(t, \vec{x})$, or even when $\vec{a}$ is a function of u, making (7) a nonlinear first-order equation.