

## Chapter V. Ordinary Differential Equations

### Outline:

- 5.1 First-order linear scalar equation.
- 5.2 High-order linear scalar equation with constant coefficients.
- 5.3 First-order linear system with constant coefficients.
- 5.4 Stability of first-order linear system.
- 5.5 Hopf Bifurcation.

We cover perturbation method next semester.

### 5.1. First-order linear scalar equation

Let us solve the problem

$$\frac{dy}{dt} + a(t)y = 0, \quad y(0) = C. \quad (1)$$

We find

$$\frac{y'}{y} = -a(t).$$

Integrate:

$$\begin{aligned} \ln y(t) - \ln y(0) &= -\int_0^t a(\tau) d\tau, \\ \ln y(t) &= \ln y(0) - \int_0^t a(\tau) d\tau, \\ y(t) &= e^{\ln y(0) - \int_0^t a(\tau) d\tau} \\ &= y(0)e^{-\int_0^t a(\tau) d\tau}. \end{aligned}$$

So the solution to (1) is

$$y(t) = Ce^{-\int_0^t a(\tau) d\tau}. \quad (2)$$

Now we consider (the first-order linear scalar equation)

$$\frac{dy}{dt} + a(t)y = f(t). \quad (3)$$

We look for a factor  $m(t)$  such that

$$m(t)y' + a(t)m(t)y = [m(t)y]'$$

From the product rule, we need  $m(t)$  to satisfy

$$m'(t) = a(t)m(t)$$

From formula (2), we find an  $m(t)$

$$m(t) = e^{\int_0^t a(s)ds}.$$

Equation (3) multiplied by  $m(t)$  becomes

$$\frac{d}{dt}[m(t)y(t)] = m(t)f(t).$$

Thus

$$m(t)y(t) - m(0)y(0) = \int_0^t m(\tau)f(\tau)d\tau,$$

or

$$y(t) = \frac{1}{m(t)}[y(0) + \int_0^t m(\tau)f(\tau)d\tau],$$

or

$$y(t) = e^{-\int_0^t a(\tau)d\tau} [y(0) + \int_0^t e^{\int_0^\tau a(s)ds} f(\tau)d\tau]. \quad (4)$$

**Examples 1.** All solutions to

$$y' + 5y = 0$$

are

$$y = ce^{-5t}.$$

**2.** All solutions to

$$y' + t^2y = 0$$

are

$$y = ce^{-\frac{1}{3}t^3}.$$

## 5.2. High-order linear scalar equations with constant coefficients

Let us solve the problem

$$\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} - 2x = 0, \quad (5)$$

$$x(0) = 0, x'(0) = 1, x''(0) = -1. \quad (6)$$

We try solutions of the form

$$x(t) = e^{\lambda t}. \quad (7)$$

Then  $x' = \lambda x(t)$ ,  $x''(t) = \lambda^2 x(t)$ ,  $x'''(t) = \lambda^3 x(t)$ . Thus  $\lambda$  needs to satisfy

$$\lambda^3 x(t) + \lambda^2 x(t) - 2x(t) = 0.$$

We find  $\lambda_1 = 1$ ,  $\lambda_2 = -1 + i$ ,  $\lambda_3 = -1 - i$ . So we have solutions  $e^t, e^{-t+it}, e^{-t-it}$ . The two complex solutions can be added or subtracted one from the other to produce two real solutions, so we have three real solutions  $e^t, e^{-t} \cos t, e^{-t} \sin t$ , since the equation is linear. Also, any linear combination of the three solutions are solutions. Thus, we have

$$x(t) = c_1 e^{-t} + c_2 e^t \cos t + c_3 e^{-t} \sin t \quad (8)$$

as the general solution formula for (5). One can use the three initial conditions (6) to determine the three coefficients  $c_1, c_2, c_3$  in (8) which we omit here.

In general the  $n$ -th-order linear scalar equation

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \dots + a_1 x' + a_0 x = 0$$

with constant coefficients  $(a_n, a_{n-1}, \dots, a_0)$  without forcing(right-hand side=0) can be solved by the guess work (7). More precisely, from the algebraic equation,

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0,$$

we can find  $n$  roots. Suppose it has  $n$  distinct roots  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Then the solution for the ODE is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots + c_n e^{\lambda_n t}.$$

If  $\lambda_1$  and  $\lambda_2$  are a pair of conjugate complex roots, say,

$$\lambda_1 = a + bi, \quad \lambda_2 = a - bi,$$

then we can replace the part  $c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$  by real solution of the form

$$\alpha e^{at} \cos(bt) + \beta e^{at} \sin(bt).$$

If  $\lambda_1$  is repeated, say  $\lambda_1 = \lambda_2$ , then  $c_2 e^{\lambda_2 t}$  is a multiple of the first solution  $c_1 e^{\lambda_1 t}$ . In this case, we replace the guess work  $e^{\lambda t}$  by  $te^{\lambda t}$ , and the solution  $c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$  is replaced by  $c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$ . If  $\lambda_1$  is repeated  $m$  times, then the solution part

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots + c_m e^{\lambda_m t}$$

is replaced by

$$c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} + \cdots + c_m t^{m-1} e^{\lambda_1 t}.$$

**Example 3.** Solve

$$\frac{d^2 x}{dt^2} - 2 \frac{dx}{dt} + x = 0.$$

*Solution* Try  $x = e^{\lambda t}$ , we find

$$\lambda^2 - 2\lambda + 1 = 0.$$

So

$$\lambda_1 = \lambda_2 = 1.$$

So the solutions are

$$x(t) = c_1 e^t + c_2 t e^t.$$