1. (5pts) Classify the critical points for the function $f(x,y) = x^3 - 6xy + 8y^3$?

Solution: The critical points (x,y) satisfy the equations

$$f_x(x,y) = 3x^2 - 6y = 0,\ f_y(x,y) = -6x + 24y^2 = 0$$

From the first equation, we have $y = x^2/2$ and substitute this into the second equation:

$$\begin{aligned} -6x + 6x^4 &= 0 \\ 6x(x-1)(x^2+x+1) &= 0 \end{aligned}$$

There are two solutions $x_1 = 0$ and $x_2 = 1$. From $y = x^2/2$, we have $y_1 = 0$ and $y_2 = 1/2$. Two critical points are $(0,0)$ and $(1,1/2)$.

To classify the critical points, we need to use the second derivative test. The second order derivatives or $f(x,y)$ are

$$f_{xx}(x,y) = 6x,\ f_{xy}(x,y) = -6,\ f_{yy}(x,y) = 48y$$

At critical point $(0,0)$, $D = f_{xx}f_{yy} - f_{xy}^2 = 0 * 0 - (-6)^2 = -36 < 0$. This point is a saddle point of $f(x,y)$.

At critical point $(1,1/2)$, $D = f_{xx}f_{yy} - f_{xy}^2 = 6 * 24 - (-6)^2 = 144 - 36 = 108 > 0$, $f_{xx} = 6 > 0$. This point is a local minimum of $f(x,y)$.

2. (5pts) Find the equation of the tangent plane and the normal line to the surface $xy + yz + xz = 3$ at the point $(1,1,1)$.

Solution 1: Use the formula for tangent planes to the level surfaces $F(x,y,z) = k$. Here $F(x,y,z) = xy + yz + xz$ is given, the partial derivatives are

$$F_x = y + z, \quad F_y = x + z, \quad F_z = x + y$$

At the point $(1,1,1)$, $F_x(1,1) = 2$, $F_y(1,1) = 2$ and $F_z(1,1) = 2$. The tangent plane equation is

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(x-x_0) + F_z(x_0,y_0,z_0)(x-x_0) = 0$$

$$2(x-1) + 2(y-1) + 2(z-1) = 0$$

$$x + y + z = 3$$

The normal line equation is

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

$$\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$$

Solution 2: Use the formula for tangent planes to the surface of $z = f(x,y)$. From the given equation $xy + yz + xz = 3$, we have $z = \frac{3-xy}{x+y}$. The partial derivatives are

$$\frac{\partial z}{\partial x} = \frac{-y^2-3}{(x+y)^2}, \quad \frac{\partial z}{\partial y} = \frac{-x^2-3}{(x+y)^2}$$

At the point $(1,1,1)$, $\frac{\partial z}{\partial x}(1,1) = -1$, $\frac{\partial z}{\partial y}(1,1) = -1$. The tangent plane equation is

$$z - z_0 = z_x(x_0,y_0)(x-x_0) + z_y(x_0,y_0)(y-y_0)$$

$$z - 1 = -(x-1) - (y-1)$$

$$x + y + z = 3$$

The direction of the normal line is $(1,1,1)$ and the equation is

$$\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$$

1. (3pts) Use Lagrange multiplier to find the maximum and minimum values of the function $f(x, y, z) = 8x - 4z$ subject to the constraint $x^2 + 10y^2 + z^2 = 5$.

Solution: The object function is $f(x, y, z) = 8x - 4z$ and the constraint function is $g(x, y, z) = x^2 + 10y^2 + z^2 = 5$. Use Lagrange multiplier we have the following equations

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g(x, y, z) &= 5 \end{aligned}$$

The exact equations are

$$8 = 2\lambda x \quad (1)$$
$$0 = 20\lambda y \quad (2)$$
$$-4 = 2\lambda z \quad (3)$$
$$x^2 + 10y^2 + z^2 = 5 \quad (4)$$

From (2) we know $y = 0$ or $\lambda = 0$. But if $\lambda = 0$ both (1) and (3) are not satisfied so we have only $y = 0$. From (1) and (3) we can get

$$x = \frac{4}{\lambda}, \quad z = \frac{-2}{\lambda}$$

Substitute into (4) we get

$$\frac{16}{\lambda^2} + \frac{4}{\lambda^2} = 5$$

We can get solutions for λ and corresponding x, z.

$$\lambda_1 = 2, \quad x_1 = 2, \quad z_1 = -1$$

$$\lambda_2 = -2, \quad x_2 = -2, \quad z_2 = 1$$

Finally we have the maximum $f(2, 0, -1) = 20$ and minimum $f(-2, 0, 1) = -20$.