1. (5 pts)Without solving the equation, determine an interval in which the solution of the given initial value problem exists. WRITE DOWN YOUR REASON.

$$(4-t^2)y' + 3(\ln t)y = \cos t, \quad y(\frac{1}{2}) = 2$$

Solution: rewrite the equation in the standard form

$$y' + \frac{3\ln t}{4-t^2}y = \frac{\cos t}{4-t^2}$$

Here $p(t) = \frac{3\ln t}{4-t^2}$ and $g(t) = \frac{\cos t}{4-t^2}$. According to theorem 2.4.1, the solution will exist on the interval where both $p(t)$ and $g(t)$ are continuous and it should include the initial t_0. Notice the continuity of $\ln t$ and $\frac{1}{4-t^2}$, $p(t)$ and $g(t)$ are defined and continuous on $(0,2)$ and $(2,\infty)$. Initial $t_0 \in (0,2)$, so (0, 2) is an interval for the solution to exist.

2. (10 pts)A tank initially contains 500 gal of fresh water. Water containing 10g per gallon salt flows into the tank at a rate of 100 gal/hr. The well stirred mixture flows out at the same rate. Let $Q(t)$ be the amount of salt in the tank at any time t.
(a) Set up and solve the initial value problem for $Q(t)$.
(b) Find the limiting concentration of salt in the current setting.
Solution(a): The differential equation is

$$\frac{dQ}{dt} = 0.01 \cdot 100 - \frac{Q}{500}100 = 1 - \frac{Q}{5}, \quad Q(0) = 0$$

The solution for this first order equation is

$$Q(t) = 5 - 5e^{-\frac{t}{5}}$$

Solution(b): The limiting salt amount is the equilibrium solution of the equation in (a). Let $\frac{dQ}{dt} = 0$, the equilibrium salt amount is 5 kg. Then the limiting salt concentration is $5/500 = 0.01kg/gal = 10g/gal$

3. (5 pts) Solve the initial value problem

$$y'' + 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Solution: The characteristic equation is

$$r^2 + 5r + 4 = 0$$

with two complex solutions conjugate to each other

$$r_1 = -1 \quad r_2 = -4$$

Then two special solution of the differential equation are

$$y_1 = e^{-t} \quad y_2 = e^{-4t}$$

and the general solution is

$$y = c_1e^{-t} + c_2e^{-4t}$$

The derivative of the general solution is

$$y' = -c_1e^{-t} - 4c_2)e^{-4t}$$

Substitute the initial conditions, we have

$$\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 - 4c_2 &= 0 \end{aligned}$$

Solve this system, we can get $c_1 = 4/3$ and $c_2 = -1/3$, the solution is

$$y = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$$