

# College Algebra

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## Chapter Four

### Polynomial & Rational Functions

# Synthetic Division

$$\begin{array}{r} 2x^3 - 1x^2 + 2x - 5 \\ \text{Divisor } x + 2 \overline{) 2x^4 + 3x^3 + 0x^2 - 1x - 5} \end{array}$$

Quotient

Dividend

$$\underline{2x^4 + 4x^3}$$

$$-1x^3 + 0x^2$$

$$\underline{-1x^3 - 2x^2}$$

$$2x^2 - 1x$$

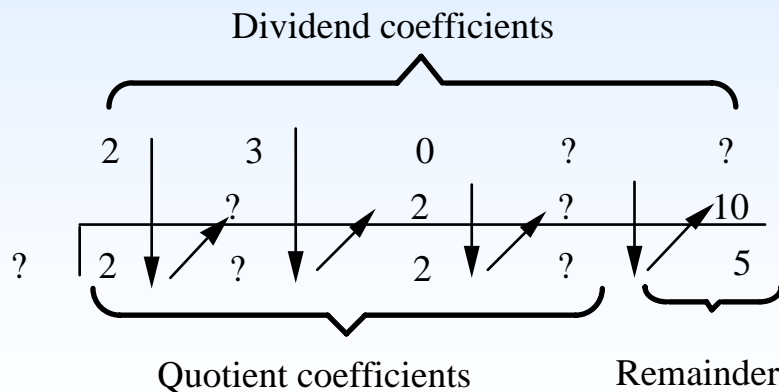
$$\underline{2x^2 + 4x}$$

$$-5x - 5$$

$$\underline{-5x - 10}$$

$$5$$

Remainder

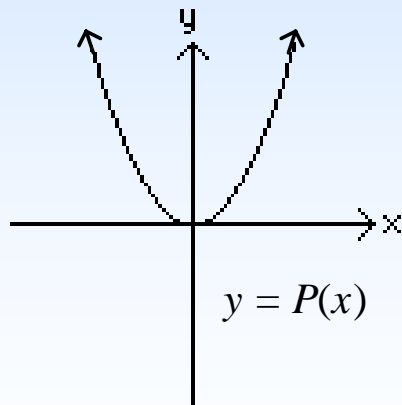


# Left and Right Behavior of a Polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

1.  $a_n > 0$  and  $n$  even

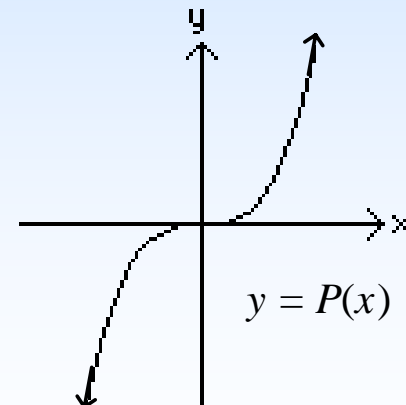
Graph of  $P(x)$  increases without bound as  $x$  decreases to the left and as  $x$  increases to the right.



$$P(x) \rightarrow \infty \text{ as } x \rightarrow -\infty$$
$$P(x) \rightarrow \infty \text{ as } x \rightarrow \infty$$

2.  $a_n > 0$  and  $n$  odd

Graph of  $P(x)$  decreases without bound as  $x$  decreases to the left and increases without bound as  $x$  increases to the right.



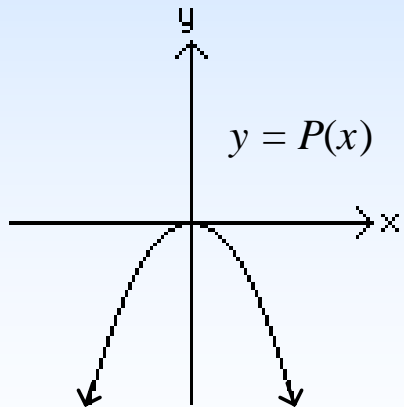
$$P(x) \rightarrow -\infty \text{ as } x \rightarrow -\infty$$
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# Left and Right Behavior of a Polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

3.  $a_n < 0$  and  $n$  even

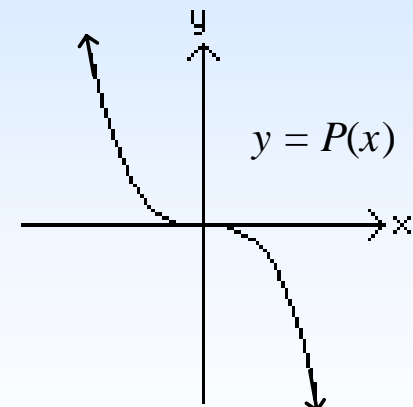
Graph of  $P(x)$  decreases without bound as  $x$  decreases to the left and as  $x$  increases to the right.



$$P(x) \rightarrow -\infty \text{ as } x \rightarrow -\infty$$
$$P(x) \rightarrow -\infty \text{ as } x \rightarrow \infty$$

4.  $a_n < 0$  and  $n$  odd

Graph of  $P(x)$  increases without bound as  $x$  decreases to the left and decreases without bound as  $x$  increases to the right.



$$P(x) \rightarrow \infty \text{ as } x \rightarrow -\infty$$
$$P(x) \rightarrow -\infty \text{ as } x \rightarrow \infty$$

# Fundamental Theorem of Algebra

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Every polynomial  $P(x)$  of degree  $n > 0$  has at least one zero.

## $n$ Zeros Theorem

Every polynomial  $P(x)$  of degree  $n > 0$  can be expressed as the product of  $n$  linear factors. Hence,  $P(x)$  has exactly  $n$  zeros—not necessarily distinct.

## Imaginary Zeros Theorem

Imaginary zeros of polynomials with real coefficients, if they exist, occur in conjugate pairs.

## Real Zeros and Odd-Degree Polynomials

A polynomial of odd degree with real coefficients always has at least one real zero.

# Strategy for Finding Rational Zeros

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Assume that  $P(x)$  is a polynomial with integer coefficients and is of degree greater than 2.

- Step 1.** List the possible rational zeros of  $P(x)$  using the rational zero theorem (Theorem 6).
- Step 2.** Construct a synthetic division table. If a rational zero  $r$  is found, stop, write

$$P(x) = (x - r)Q(x)$$

and immediately proceed to find the rational zeros for  $Q(x)$ , the reduced polynomial relative to  $P(x)$ . If the degree of  $Q(x)$  is greater than 2, return to step 1 using  $Q(x)$  in place of  $P(x)$ . If  $Q(x)$  is quadratic, find all its zeros using standard methods for solving quadratic equations.

# Location Theorem

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If  $f$  is continuous on an interval  $I$ ,  $a$  and  $b$  are two numbers in  $I$ , and  $f(a)$  and  $f(b)$  are of opposite sign, then there is at least one  $x$  intercept between  $a$  and  $b$ .

## Upper and Lower Bounds of Real Zeros

Given an  $n$ th-degree polynomial  $P(x)$  with real coefficients,  $n > 0$ ,  $a_n > 0$ , and  $P(x)$  divided by  $x - r$  using synthetic division:

- 1. Upper Bound.** If  $r > 0$  and all numbers in the quotient row of the synthetic division, including the remainder, are nonnegative, then  $r$  is an upper bound of the real zeros of  $P(x)$ .
- 2. Lower Bound.** If  $r < 0$  and all numbers in the quotient row of the synthetic division, including the remainder, alternate in sign, then  $r$  is a lower bound of the real zeros of  $P(x)$ .

**[Note:** In the lower-bound test, if 0 appears in one or more places in the quotient row, including the remainder, the sign in front of it can be considered either positive or negative, but not both. For example, the numbers 1, 0, 1 can be considered to alternate in sign, while 1, 0, -1 cannot.]

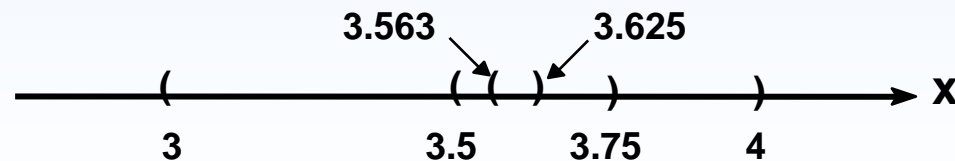
# The Bisection Method

Approximate to one decimal place the zero of

$$P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90$$

on the interval (3, 4).

Sign Change Interval (a, b)	Midpoint m	Sign of P		
		P(a)	P(m)	P(b)
(3, 4)	3.5	-	-	+
(3.5, 4)	3.75	-	+	+
(3.5, 3.75)	3.625	-	+	+
(3.5, 3.625)	3.563	-	-	+
(3.563, 3.625)	Stop here			



Nested intervals produced by the Bisection Method

# Synthetic Division on a Graphing Utility

## Program SNYDIV

TI-82/TI-83

```
Lbl A
Prompt R
2→I
dim(L1)→N
{0}→L2
N→dim(L2)
L1(I)→L2(I)
Lbl B
L2(I-1)*R+L1(I)→L2(I)
1+I→I
If I≤N
Goto B
Pause L2
Goto A
```

TI-85/TI-86

```
Lbl A
Prompt R
2→I
dimL L1→N
{0}→L2
N→dimL L1
L1(I)→L2(I)
Lbl B
L2(I-1)*R+L1(I)→L2(I)
1+I→I
If I≤N
Goto B
Pause L2
Goto A
```

## OUTPUT

```
{1, -2, -10, 40, -90}→L1
{1 -2 -10 40 -90}
SYNDIV
R=?1
{1 -1 -11 29 -61}
R=?2
{1 0 -10 20 -50}
R=?3
{1 1 -7 19 -33}
R=?4
{1 2 -2 32 38}
R=?5
{1 3 5 65 235}
R=? -1
{1 -3 -7 47 -137}
R=? -2
{1 -4 -2 44 -178}
R=? -3
{1 -5 5 25 -165}
R=? -4
{1 -6 14 -16 -26}
R=? -5
{1 -7 25 -85 335}
R=?
```

# Graphing a Rational Function:

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$$f(x) = \frac{n(x)}{d(x)}$$

- Step 1.** *Intercepts.* Find the real solutions of the equation  $n(x) = 0$  and use these solutions to plot any  $x$  intercepts of the graph of  $f$ . Evaluate  $f(0)$ , if it exists, and plot the  $y$  intercept.
- Step 2.** *Vertical Asymptotes.* Find the real solutions of the equation  $d(x) = 0$  and use these solutions to determine the domain of  $f$ , the points of discontinuity, and the vertical asymptotes. Sketch any vertical asymptotes as dashed lines.
- Step 3.** *Sign Chart.* Construct a sign chart for  $f$  and use it to determine the behavior of the graph near each vertical asymptote.
- Step 4.** *Horizontal Asymptotes.* Determine whether there is a horizontal asymptote and if so, sketch it as a dashed line.
- Step 5.** *Symmetry.* Determine symmetry with respect to the vertical axis and the origin.
- Step 6.** *Complete the Sketch.* Complete the sketch of the graph by plotting additional points and joining these points with a smooth continuous curve over each interval in the domain of  $f$ . Do not cross any points of discontinuity.

# Partial Fraction Decomposition

Any proper fraction  $P(x)/D(x)$  reduced to lowest terms can be decomposed in the sum of partial fractions as follows:

1. If  $D(x)$  has a nonrepeating linear factor of the form  $ax + b$ , then the partial fraction decomposition of  $P(x)/D(x)$  contains a term of the form

$$\frac{A}{ax + b} \quad A \text{ a constant}$$

2. If  $D(x)$  has a  $k$ -repeating linear factor of the form  $(ax + b)^k$ , then the partial fraction decomposition of  $P(x)/D(x)$  contains  $k$  terms of the form

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \dots + \frac{A_k}{(ax + b)^k} \quad A_1, A_2, \dots, A_k \text{ constants}$$

3. If  $D(x)$  has a nonrepeating quadratic factor of the form  $ax^2 + bx + c$ , which is prime relative to the real numbers, then the partial fraction decomposition of  $P(x)/D(x)$  contains a term of the form

$$\frac{Ax + B}{ax^2 + bx + c} \quad A, B \text{ constants}$$

4. If  $D(x)$  has a  $k$ -repeating quadratic factor of the form  $(ax^2 + bx + c)^k$ , where  $ax^2 + bx + c$  is prime relative to the real numbers, then the partial fraction decomposition of  $P(x)/D(x)$  contains  $k$  terms of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k} \\ A_1, \dots, A_k, B_1, \dots, B_k \text{ constants}$$