2

THE PYTHAGOREAN THEOREM

2.1 Introduction

Here is a fact seemingly not worth mentioning for its triviality: **Still water in a resting container, with no disturbances, shall remain at rest.** I think it is remarkable that this fact has the Pythagorean theorem as a corollary (p. 17). In addition, this seeming triviality implies the law of sines (p. 18), the Archimedian buoyancy law, and the 3D area version of the Pythagorean theorem (p. 19).

The proof of the Pythagorean theorem, described in section 2.2, suggested a kinematic proof of the Pythagorean theorem, described in section 2.6. The motion-based approach makes some other topics very transparent, including

- The fundamental theorem of calculus.
- The computational formula for the determinant.
- The expansion of the determinant in a row.

All these are described in this chapter.

Several more physical proofs of the Pythagorean theorem are given here, one using springs, and the other using kinetic energy.

The unifying theme of this chapter is the Pythagorean theorem, although we do go off on a few short tangents.

2.2 The "Fish Tank" Proof of the Pythagorean Theorem

Let us build a prism-shaped "fish tank" with our right triangle as the base (figure 2.1). We mount the tank so that it can rotate freely

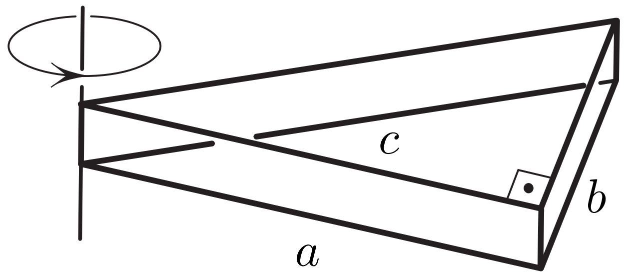

Figure 2.1. The water-filled fish tank, free to rotate around a vertical edge, has no desire to.

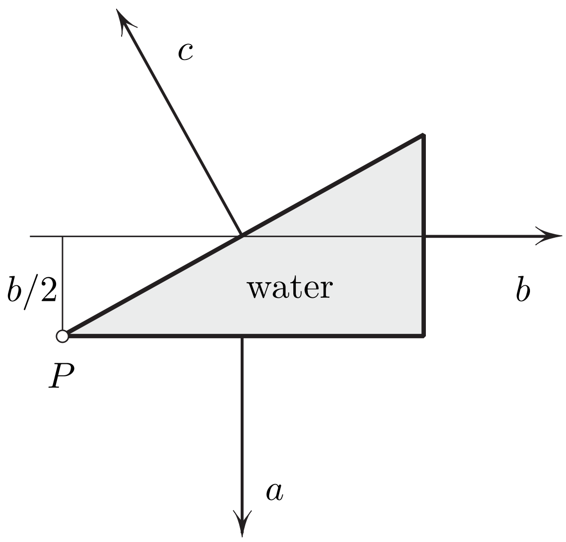

Figure 2.2. The Pythagorean theorem is equivalent to the vanishing of the combined torque upon the tank around *P*.

around the vertical axis through one end of the hypotenuse. Now let us fill our fish tank with water.

The water pushes on the walls in three competing directions as figure 2.2 shows, each force trying to rotate the tank around P. Of course, the competition is a draw: the tank has zero desire to rotate. Otherwise we would have had an engine which uses no fuel—a so-called perpetual motion machine, forbidden by the law of conservation of energy.

In this case the "desire" is the sum of the three torques of the pressure forces. We note here[1] that the torque of the force around a pivot point P is simply the force's magnitude times the distance from the line of force to the pivot point. The torque measures the intensity with which the force tries to rotate the object it's applied to around P.

For convenience, let us assume the force of pressure to be 1 pound per unit length of the wall—we can always achieve it by adjusting water depth. The three forces are then a, b, and c; the corresponding levers are $a/2$, $b/2$, and $c/2$, and the zero torque condition reads

$$a \cdot \frac{a}{2} + b \cdot \frac{b}{2} - c \cdot \frac{c}{2} = 0, \tag{2.1}$$

or $a^2 + b^2 = c^2$, giving us the Pythagorean theorem!

Still water. Note that we didn't have to build the fish tank, not even in the thought experiment; rather, we can imagine the prism of water embedded in a larger body of water. The Pythagorean theorem follows as before from the fact that the prism will not spontaneously rotate under the pressure of the surrounding fluid on its vertical faces. We conclude that the Pythagorean theorem is a consequence of the fact that still water remains still.

Exercise. *From a point A outside a circle draw a tangent line AT and a secant line APQ as shown in figure 2.3. Prove that*

$$AP \cdot AQ = AT^2. \tag{2.2}$$

Hint: Consider the shaded curvilinear triangle APT in figure 2.3, thought of as a rigid container filled with gas and allowed to pivot around O.
As explained in section 2.3 in a different context, (2.2) expresses the fact that the shaded area remains unchanged under rotations around O. Similarly, the Pythagorean theorem expresses the fact that the area of a right triangle remains unchanged as the triangle is rotated around one of the ends of the hypotenuse.

[1] See section A.5 for full background.

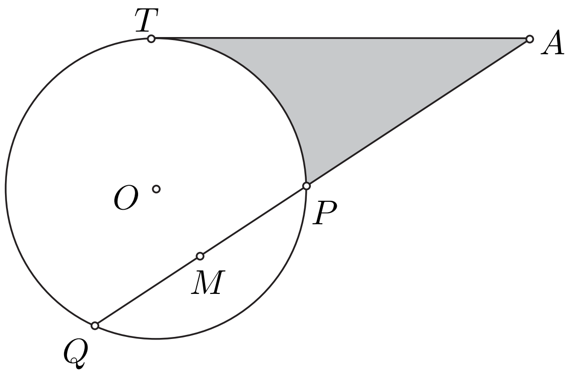

Figure 2.3. Proving $AP \cdot AQ = AT^2$.

2.3 Converting a Physical Argument into a Rigorous Proof

The pivotal[2] point of the "fish tank" proof of the Pythagorean theorem was the vanishing of the net torque around P (figure 2.1). How can we restate this zero-torque idea in purely mathematical terms, without appealing to physical concepts? Here is the answer.

The *physical* statement (2.1) of zero net torque around P translates into the *geometrical* statement that the area of the triangle does not change when the triangle is rotated around P.[3] Here is the proof of this equivalence.

Let $A(\theta)$ be the area of the triangle rotated around P through the angle θ. This area is, of course, independent of θ:

$$A'(\theta) = 0,$$

and we claim that it is this constancy of the area that is equivalent to the zero-torque condition (2.1). To show this equivalence it suffices to show that

$$A'(\theta) = a \cdot \frac{a}{2} + b \cdot \frac{b}{2} - c \cdot \frac{c}{2}. \tag{2.3}$$

[2]This pun was not originally intended.

[3]Here is an example where a trivial-sounding fact (the area of the triangle doesn't change under rotations) hides something less trivial (the Pythagorean theorem.)

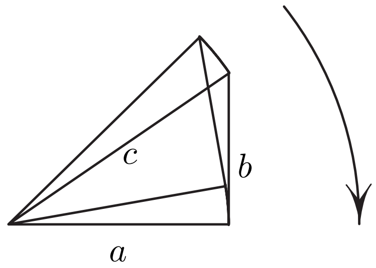

Figure 2.4. The area swept by the two legs equals the area swept by the hypotenuse.

To demonstrate (2.3) we rotate the triangle through a small angle $\Delta\theta$ around P. The side a sweeps a sector of area $\frac{1}{2}a^2\Delta\theta$, with a similar expression for c. In fact, the area swept by b is given by the same expression: $\frac{1}{2}b^2\Delta\theta$. Indeed, b executes two motions simultaneously: (i) sliding in its own direction, contributing nothing to the rate of sweeping of the area, and (ii) rotation around its leading end. We conclude that the area swept is $\frac{1}{2}b^2\Delta\theta$. The total area swept by all three sides is

$$\Delta A = \left(\frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{2}c^2\right)\Delta\theta.$$

Here the minus sign is due to the fact that the area is "lost" through the hypotenuse. Dividing by $\Delta\theta$ and taking the limit as $\Delta\theta \to 0$, we obtain (2.3).

Here are a few other applications of the idea of sweeping:

1. A "ring" proof of the Pythagorean theorem described in section 2.6.
2. A remark on the area between the tracks of two wheels of a bike (section 6.1).
3. A visual proof that the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ equals the area of a parallelogram generated by the vectors $\langle a, c\rangle$ and $\langle b, d\rangle$ (section 2.5).
4. A visual proof of the formula for the row decomposition of a determinant (section 2.5).