

BOOK FOUR

Apollonius greets Attalus ¹.

Earlier, I presented the first three books of my eight books treatise on conics to Eudemus of Pergamum, but with his having passed away I decided to write out the remaining books for you, because of your earnest desire to have them. To start, then, I am sending you the fourth book. This book treats of the greatest number of points at which sections of a cone can meet one another or meet a circumference of a circle, assuming that these do not completely coincide, and, moreover, the greatest number of points at which a section of a cone or a circumference of a circle can meet the opposite hyperbolas. Besides these questions, there are more that a few others of a similar character Conon of Samos presented the first mentioned question to Thrasydaeus without giving a correct proof, for which he was rightly attacked by Nicoteles of Cyrene ². As for the second question, Nicoteles, in replying to Conon only mentions that it can be proved, but I have found no proof either by him or by anyone else. Regarding the third and similar questions, however, I have not found them even noticed by anyone. And all these things just spoken of, whose demonstrations I have not found anywhere, require many and various striking theorems, of which most happen to be presented in the first three books of my treatise on conics, and the rest in this book. The investigation of these theorems is also of considerable use in the synthesis of problems and limits of possibility. So, Nicoteles was not speaking truly when, for the sake of his argument with Conon, he said that none of the things discovered by Conon were of any use for limits of possibility, but even if the limits of possibility are able to be obtained completely without these things yet, surely, some matters are more readily perceived by means of them, for example, whether a problem might be done in many ways, and in how many ways, or again, whether it might not be done at all. Moreover, this preliminary knowledge brings with it a solid starting point for investigations, and the theorems are useful for the analysis of limits of possibility. But apart from such usefulness, these things are also worthy of acceptance for the demonstrations themselves: indeed, we accept many things in mathematics for this and no other reason.

[Proposition] 1

If a point is taken outside a section of a cone or the circumference of a circle, and from this point two straight lines are drawn towards the section, of which one touches the section and other cuts the section at two points, and if the straight line cut off inside the section is divided in that ratio which the whole straight line cut off has to the part outside bounded between the point and the section, so that homologous straight lines are at the same point, then the straight line drawn from the point of contact to the point of division will meet the line of the section, and the straight line drawn from the point of meeting to outside point will touch the section ³.

Let there be the section of a cone or the circumference of a circle $AB\Gamma$ and let Δ be taken outside the section, from Δ let ΔB touch the section at B and let $\Delta E\Gamma$ cut the section at E and Γ , and let as ΓZ is to ZE , so $\Gamma\Delta$ is to ΔE .

I say that the straight line from B to Z will meet the section, and the straight line drawn from the point of meeting to Δ will touch the section.

[Proof]. For let ΔA be drawn from Δ touching the section, and let BA be joined cutting $E\Gamma$, if possible, not at Z , but at H . Now since $B\Delta$ and ΔA , touch the section, BA is drawn from the point of contact, and $\Gamma\Delta$ goes through AB cutting the section at Γ and E and meeting AB at H , [according to Proposition III.37] as $\Gamma\Delta$ is to ΔE , so ΓH is to HE . But this is impossible for it was assumed that as $\Gamma\Delta$ is to ΔE , so ΓZ is to ZE . Therefore BA does not cut ΓE at a different point from Z , therefore it cuts ΓE at Z .

[Proposition] 2

This is proved for all sections together. However regarding the hyperbola only, if ΔE touches the hyperbola and $\Delta\Gamma$ cuts it at two points E and Γ , and if the point of contact, B , is between E and Γ , and Δ is inside the angle between the asymptotes, then the proof is carried out similarly for from Δ it is possible to draw another straight line ΔA touching the hyperbola and the rest of the proof is done similarly ⁴.

[Proposition] 3

With the same suppositions if E and Γ do not contain the point of contact, B , between them, and let Δ be inside the angle between the asymptotes. Therefore from Δ it is possible to draw another straight line ΔA touching the section, and rest is proved as before ⁵.

[Proposition] 4

With the same suppositions if the points of the meeting E and Γ contain the point of contact, B , and Δ is in the angle adjacent to the angle between the asymptotes, then the straight line from the point of contact to the point of division meets the opposite hyperbola, and the straight line drawn from the point of meeting to Δ will touch the opposite hyperbola ⁶.

[Proof]. For let B and Θ be opposite hyperbolas, let $K\Lambda$ and $M\Xi N$ be asymptotes, and let Δ be in the angle $\Lambda\Xi N$. Furthermore let ΔB be drawn from Δ touching, and $\Delta\Gamma$ cut one of the hyperbolas, let the points of meeting E and Γ contain the point of contact B , and let as ΓZ is to ZE , so $\Gamma\Delta$ is to ΔE . It is to be shown that the straight line joined from B to Z will meet the hyperbola Θ , and that the straight line from the point of meeting to Δ will touch the hyperbola B .

Let $\Delta\Theta$ be drawn from Δ touching the hyperbola, and let the straight line ΘB all fall, if possible, not at Z , but at H . Therefore [according to Proposition III.37] as $\Gamma\Delta$ is to ΔE , so ΓH is to HE . But it is impossible for it was assumed that as $\Gamma\Delta$ is to ΔE , so ΓZ is to ZE .

[Proposition] 5

With the same supposition if Δ is on an asymptote, the straight line drawn from B to Z will be parallel to the asymptote ⁷.

[Proof]. For let the same be supposed, let Δ be on one of the asymptotes, MN. It is to be shown that the straight line drawn from B parallel to MN will fall on Z. For if not, let the straight line, if possible, be BH. But then [according to Proposition III.35] as $\Gamma\Delta$ is to ΔE , so ΓH is to HE, but it is impossible.

[Proposition] 6

If a point is taken outside a hyperbola, and from this point two straight lines are drawn to the hyperbola, one of which touches the hyperbola, and the other is parallel to one of the asymptotes, and if the segment of the latter straight line inside the hyperbola is equal to the segment cut off between the hyperbola and the point, then the straight line joined from the point of contact of the former straight line to the taken point will meet the hyperbola, and the straight line drawn from the point of meeting to the point outside will touch the hyperbola ⁸.

Let there be the hyperbola AEB, let Δ be some point taken outside it, and, to start, let Δ be inside the angle between the asymptotes, and from Δ let B Δ be drawn touching the hyperbola, let ΔEZ be parallel to the other of the asymptotes, and let EZ be equal to ΔE .

I say that the straight line joining from B and Z will meet the hyperbola and the straight line from the point of meeting to Δ will touch the hyperbola.

[Proof]. For let ΔA be drawn touching the hyperbola, and let BA be joined and cutting ΔE , if possible, not at Z but at some other point H. Then [according to Proposition III.30] ΔE will be equal to EH. But it is impossible for it was assumed that ΔE is equal to EZ.

[Proposition] 7

With the same suppositions Δ be in the angle adjacent to the angle between the asymptotes.

I say that the same will come to pass ⁹.

[Proof]. For let $\Delta\Theta$ be drawn touching the hyperbola and let ΘB be joined and let, if possible, fall not on Z but on H. Therefore [according to Proposition III.31] ΔE is equal to EH. But it is impossible for it was assumed that ΔE is equal to EZ.

[Proposition] 8

With the same suppositions if Δ is on one of the asymptotes and let the remaining constructions be the same.

I say that the straight line drawn from the point of contact to the end of the straight line cut off will be parallel to the asymptote on which Δ is situated ¹⁰.

[Proof]. Let there be the construction just mentioned, and let EZ be equal to ΔE , and from B let BH be drawn, if possible, parallel to MN. Therefore [according to Proposition III.34] ΔE is equal to EH. But it is impossible for it was assumed that ΔE is equal to EZ.

[Proposition] 9

If from the some point two straight lines are drawn each cutting a section of a cone or the circumference of a circle at two points ,and if the segments cut off inside are divided in the same ratio as the wholes are to the segments cut off outside, so that the homologous straight lines are at the same point, then the straight line drawn through the points of division will meet the section at two points, and straight lines drawn from the points of meeting to the point outside will touch the section ¹¹.

Let there be the section described by us ΛB , and from a point Δ [outside it] let ΔE and ΔZ be drawn cutting the section at Θ and E and at Z and H , respectively. Furthermore let as $E\Lambda$ is to $\Lambda\Theta$, so ΔE is to $\Theta\Delta$, and at ZK is to KH , so ΔZ is to ΔH .

I say that the straight line joining Λ to K will meet the section at both ends, and the straight lines joining the points of meeting will touch the section.

[Proof]. For since $E\Delta$ and $Z\Delta$ both cut the section at two points, it is possible to draw a diameter of the section through Δ , and with that also straight lines touching the section on either side. Let straight lines ΔB and ΔA be drawn touching section, and let BA be joined not passing through ΛK , if possible, but through only one of these two, or through neither. First, let it pass through Λ only and let it cut ZH at M . Therefore [according to Proposition III.37] as $Z\Delta$ is to ΔH , so ZM is to MH , but this is impossible for it has been assumed that as $Z\Delta$ is to ΔH , so ZK is to KH .

If BA passes through neither Λ nor K then, the absurdity occurs with regards to each straight line ΔE and ΔZ .

[Proposition] 10

The reasons above are common for all sections. However regarding the hyperbola only, if the other reasons are assumed, and if the points of meeting of the one straight line are between the points of meeting of the other straight line, and if Δ is inside the angle between the asymptotes, the same reasons said above will happen as we said above in Theorem 2 [Proposition IV.2] ¹².

[Proposition] 11

With the same suppositions if the points of meeting of one of the straight lines do not contain the points of meeting of the other straight line, then Δ is in the angle between the asymptotes and the diagram and the proof will be the same as in Theorem 9 [Proposition IV.9] ¹³.

[Proposition] 12

With the same suppositions if the points of meeting of one of the straight lines contain those other straight lines, and if the chosen point is in the angle adjacent to

the angle between the asymptotes, then the straight line drawn through the points of division and continued will meet the opposite hyperbola, and the lines drawn from the points of meeting to Δ will touch the opposite hyperbolas ¹⁴.

Let there be the hyperbola ZH , and its asymptotes NE and $O\Pi$, and its center be Π . Furthermore let Δ be in the angle $\Xi P\Pi$, let ΔE and ΔZ be drawn cutting the hyperbola each at two points, let E and Θ be between Z and H , and let be that $E\Delta$ is to $\Delta\Theta$, so $E\kappa$ is to $\kappa\Theta$, and that as $Z\Delta$ is to ΔH , so $Z\Lambda$ is to ΛH .

It is to be shown that the [straight line] through κ and Λ will meet both [the hyperbola] EZ and also the opposite hyperbola, and the lines from the points of meeting to Δ will touch the hyperbolas.

[Proof]. For let M be the opposite hyperbola, and from Δ let ΔM and $\Delta\Sigma$ be drawn touching the hyperbola, let $M\Sigma$ be joined, and, if possible, let it not pass through κ and Λ , but rather through only one of these two points for through neither.

First let it pass through κ and cut ZH at X . Therefore [according to Proposition III.37] as $Z\Delta$ is to ΔH , so XZ is to XH . But this is impossible for it has been assumed that as $Z\Delta$ is to ΔH , so $Z\Lambda$ is to ΛH .

If $M\Sigma$ passes through neither κ nor Λ , then the impossibility occurs with regards to each straight line $E\Delta$ and ΔZ .

[Proposition] 13

With the same suppositions if Δ is on one of the asymptotes, and the remaining constructions are assumed to be the same, then the straight line drawn through the points of division will be parallel to the asymptote on which the point is situated and continued will meet the hyperbola. Moreover the straight line drawn from the point of meeting to the point situated on the asymptote will touch the section ¹⁵.

Let there be a hyperbola and its asymptotes, and let Δ be taken on one of the asymptotes. Let straight lines be drawn and divided as we have said above, and let a straight line ΔB be drawn from Δ touching the hyperbola.

I say that the straight line drawn from B parallel to ΠO passes through κ and Λ .

[Proof]. For let if not so, then surely it will pass through one of these points for two neither.

Let it pass through κ only, therefore [according to Proposition III.35] as $Z\Delta$ is to ΔH , so $Z\kappa$ is to κH . But it is impossible. Therefore the straight line drawn through B parallel to ΠO will not pass through κ only. Therefore it will pass through both points [κ and Λ].

[Proposition] 14

In the same suppositions if Δ is on one of the asymptotes, and ΔE cuts the hyperbola at two points, and ΔH parallel to the other asymptote cuts the hyperbola at H only, and if as ΔE is to $\Delta\Theta$, so $E\kappa$ is to $\kappa\Theta$, and $H\Lambda$ is equal to ΔH is situated in a straight line with ΔH , then the straight line drawn through κ and Λ will be parallel to the asymptote, and

will meet the hyperbola, and the straight line drawn from the point of meeting to Δ will touch the hyperbola for similarity to what was said above, ΔB will touch the hyperbola.

I say that the straight line drawn from B parallel to the asymptote ΠO will pass through K and Λ .

[Proof]. Indeed, if it passed through K only, ΔH will not be equal to $H\Lambda$ [according to Proposition III.34], which is impossible. And if it passes through Λ only then it will not be that [according to Proposition III.35] as $E\Delta$ is to $\Delta\Theta$, so EK is to $K\Theta$, and if it passed neither through K nor through Λ , the impossibility will occur in both ways. Therefore it will pass through both points.

[Proposition] 15

If in opposite hyperbolas a point is taken between two hyperbolas, and if a straight line from this point touches one of opposite hyperbolas, and another straight line cuts each of opposite hyperbolas, and if as the straight line between the point and the one hyperbola which the first straight line does not touch is to the straight line between the point and the other hyperbola, so the greater straight line between the hyperbolas is to its excess over the latter, set in a straight line with it and with the homologous lines being at the same ends, then the straight line drawn from the end of the greater straight line to the point of contact will meet the section, and the straight line drawn from the point of meeting to the taken point will touch the section ¹⁷.

Let there be the opposite hyperbolas A and B and let some point Δ be taken between the hyperbolas and in the angle between the asymptotes, and from this point let ΔZ be drawn touching the section and $A\Delta B$ be drawn cutting the section. Furthermore as $A\Gamma$ is to ΓB , so $A\Delta$ is to ΔB . It is to be shown that the straight line drawn from Z to Γ will meet the section, and the straight line drawn from the point of meeting to Δ will touch the section.

[Proof]. For let since Δ is situated in the angle containing the section, it is possible to draw from Δ another straight line touching the section [according to Proposition II.49]. Let ΔE be drawn, let ZE be drawn and let it pass, if possible, not through Γ , but through H. It will then [according to Proposition III.37] that as $A\Delta$ is to ΔB , so AH will be to HB , which is impossible for it was assumed that as $A\Delta$ is to ΔB , so $A\Gamma$ is to ΓB .

[Proposition] 16

If Δ is situated in the angle adjacent to the angle between the asymptotes, and let the remaining construction be the same ¹⁸.

I say that the straight line joining Z to Γ will then continued to meet the opposite hyperbola, and the straight line from the point of meeting to Δ will touch the opposite hyperbola.

[Proof]. For let the same reason be as before, and let Δ be in the angle adjacent to the angle between the asymptotes, and let ΔE be drawn from Δ touching the hyperbola A, let EZ be joined and when continued let it not pass through Γ , but through H, if possible. Then it will be that [according to Proposition III.39] as AH is to HB , so $A\Delta$ will be to ΔB , which is impossible for it was assumed that as $A\Delta$ is to ΔB , so $A\Gamma$ is to ΓB .

[Proposition] 17

With the same suppositions let Δ be on an asymptote ¹⁹.

I say that the straight line drawn from Z to Γ will be parallel to the asymptote on which Δ is situated.

Let there be the same as before, let Δ be on one of asymptotes let a straight line be drawn through Z parallel to the asymptote, and, if possible, let it not fall on Γ but on H . It will then be [according to Proposition III.36] as $A\Delta$ is to ΔB , so AH will be to HB , which possible. Therefore the straight line from Z parallel to the asymptote will fall on Γ .

[Proposition] 18

If in opposite hyperbolas a point is taken between the hyperbolas and from this point two straight lines are drawn cutting each of hyperbolas, and if as the straight lines between one of hyperbolas and the point are two those between the other hyperbola and the same point, so are straight lines greater than those cut off between the opposite hyperbola to their excess over the latter, then the straight line drawn through the ends of the greater straight lines will meet the hyperbolas, and the straight lines drawn from the points of meeting to the original taken point will touch the hyperbolas ²⁰.

Let there be the opposite hyperbolas A and B , and let Δ be between the hyperbolas. Let it be assumed first that Δ be in the angle between the asymptotes, and through Δ let $A\Delta B$, $\Gamma\Delta\Theta$ be drawn. $A\Delta$ is greater than ΔB , and $\Gamma\Delta$ is greater than $\Delta\Theta$ since [according to Proposition II.16] BN is equal to AM .

Furthermore let as AK is to KB , so $A\Delta$ is to ΔB , and let as ΓH is to $H\Theta$, so $\Gamma\Delta$ is to $\Delta\Theta$.

I say that the straight line through K and H meets the hyperbolas, and the straight lines from Δ to the points of meeting will touch the section.

[Proof]. For since Δ is inside of the angle between the asymptotes, it is possible to draw two straight lines touching the section [according to Proposition II.49]. Let ΔE and ΔZ be drawn, and let EZ be joined. It will, thus, pass through K and H for if it passes through one of these points only the other straight line will be cut in the same ratio by another point, which is impossible. If it passes through neither point, the same impossibility will occur in both straight lines.

[Proposition] 19

Let Δ be taken then in the angle adjacent to the angle between the asymptotes and let straight lines be drawn cutting the section and divided as said above ²¹.

I say that the straight line drawn through K and H will meet each of opposite hyperbolas, and the straight lines from the point of meeting to Δ will touch the section ²¹.

[Proof]. For let ΔE and ΔZ be drawn from Δ touching each of the hyperbolas. Therefore the straight line through E and Z will pass through K and H for if not so, it

will surely go through one of two, or through neither, and again one will similarly inter from this an absurdity.

[Proposition] 20

If the point is taken on an asymptote, and the remaining constructions are the same, then the straight line drawn through the ends of the greater straight lines will be parallel to the asymptote on which the point is situated, and the straight line drawn from the point of meeting of the section and the straight line drawn through the ends of the greater straight lines will touch the section ²².

Let there be the opposite hyperbolas A and B, and let Δ be on one of the asymptotes, and let the remaining construction be the same.

I say that the straight line through K and H meets the section, and the straight line from the point of meeting to Δ will touch the section.

[Proof]. For let ΔZ be drawn from Δ touching the section, and a straight line be drawn from Z parallel to the asymptote on which Δ is situated, it will then pass through K and H for if not so, it will either pass through one of two or neither, and the same impossibilities will occur as before [according to Proposition III.36]

[Proposition] 21

Again let there be the opposite hyperbolas A and B, and let Δ be on one of the asymptotes, let ΔBK be parallel to one of two asymptotes, meet the section at one point B only, but let $\Gamma\Delta\Theta$ meet both of hyperbolas.

Furthermore let as ΓH be to $H\Theta$, so $\Gamma\Delta$ be to $\Delta\Theta$, and let ΔB be equal to BK .

I say that the straight line through K and H will meet the section and will be parallel to the asymptote on which Δ is situated, and that the straight line drawn from the point of meeting to Δ will touch the section ²³.

[Proof]. For let ΔZ be drawn touching the section, and let a straight line be drawn parallel to the asymptote on which Δ is situated. It will thus pass through K and H for if not so, the absurdity said before will occur [according to Proposition III.36]

[Proposition] 22

Similarly, let there be the opposite hyperbolas and their asymptotes, and let Δ be similarly taken. Let $\Gamma\Delta\Theta$ be taken cutting the hyperbolas, and ΔB be taken parallel to one of two asymptotes.

Moreover as $\Gamma\Delta$ is to $\Delta\Theta$, let ΓH be to $H\Theta$, and let BK be equal to ΔB .

I say that the straight line through K and H will meet each of the opposite hyperbolas, and the straight lines from the points of meeting to Δ will touch the section ²⁴.

[Proof]. For let ΔE and ΔZ be drawn touching the section, let EZ be joined, and, if possible, let it not pass through K and H, but through one of these two points or

neither. If, on the one hand, it passes through H only, ΔB will not be equal to BK, but to some other straight line which [according to Proposition III.31] is impossible. If, on the other hand, it passed through K only, it will not be that as $\Gamma\Delta$ is to $\Delta\Theta$, so ΓH is to $H\Theta$, but, some straight line to some other straight line [according to Proposition III.36]. If yet it passes through neither of K and H, then both impossibilities will occur.

[Proposition] 23

Again let there be the opposite hyperbolas A and B, and let Δ be in the angle adjacent to the angle between the asymptotes. Let $B\Delta$ be drawn cutting the hyperbola B at one point only, and thus parallel to one of two asymptotes, and let ΔA be drawn similarly to the hyperbola A, and let ΔB be equal to BH and ΔA to AK.

I say that the straight line through K and H meets the hyperbolas and the straight lines drawn from the points of meeting to Δ will touch the hyperbolas.

[Proof]. For let ΔE and ΔZ be drawn touching the hyperbolas, let EZ be joined, and, if possible, let it not pass through KH. So, either it will pass through one of these two points or through neither of them, and either ΔA will not be equal to AK, but some other straight line, which is impossible, or ΔB will not be equal to BH, or neither will be equal to neither, and again the same impossibility will occur in both cases [according to Proposition III.31]. Therefore EZ will pass through K and H.

[Proposition] 24

A section of a cone will not meet a section of a cone or the circumference of a circle in such way that a part of them will be the same and another part will not be common ²⁶.

[Proof]. For let, if possible, let the section of a cone $\Delta AB\Gamma$ meet [other section of a cone or] the circumference of the circle $EAB\Gamma$, let the same part $AB\Gamma$ of these sections be common and let $A\Delta$ and AE not be common. Let Θ be taken on this part, let ΘA be joined, and through an arbitrary point E draw $\Delta E\Gamma$ parallel to $A\Theta$. Moreover bisect $A\Theta$ at H, and through H draw the diameter BHZ. Therefore the straight line through B parallel to $A\Theta$ touches each of the sections, and also will be parallel to $\Delta E\Gamma$. Also in one section ΔZ will be equal to $Z\Gamma$, and in other section [according to Propositions I.46 and I.47] EZ will be equal to $Z\Gamma$, so that also ΔZ and ZE are equal, but this is impossible ²⁷.

[Proposition] 25

A section of a cone does not cut a section of a cone or the circumference of a circle at more than four points ²⁸.

[Proof]. For let, if possible, them cut at five points A, B, Γ , Δ , E, and let the points of meeting A, B, Γ , Δ , E be taken in succession so the no point of meeting between them is left out, and let AB and $\Gamma\Delta$ be joined and continued. So, these straight lines will meet out side the section in the cases of the parabola and the hyperbola [ac-

cording to Propositions II.24 and II.25]. Let them meet at Λ , and let as $A\Lambda$ be to ΛB , so AO be to OB , and as $\Delta\Lambda$ be to $\Lambda\Gamma$, so $\Delta\Pi$ be to $\Pi\Gamma$.

Therefore the straight line from Π to O joined and continued will meet the section on each side and the straight lines joining the points of meeting and Λ [according to Proposition IV.9] will touch the section. Let the points of contact are Θ and P and let $\Theta\Lambda$ and ΛP be joined. Hence they touch the section.

Therefore since there is no point of meeting between B and Γ the straight line $E\Lambda$ cuts each of the sections. Let it cut them at M and H . Therefore in one hyperbola as EN is to NH , so $E\Lambda$ is to ΛH , and in the other hyperbola as EN is to NM , so $E\Lambda$ is to ΛM . But it is impossible, so that also what was assumed at the start is impossible.

If AB and $\Delta\Gamma$ are parallel, the sections will, of course, be the ellipses or the circumference of a circle. Let AB and $\Gamma\Delta$ be bisected at O and Π , and let $O\Pi$ be joined and continued on each side. Then it will meet the sections. So let it meet them at Θ and P . Then ΘP will be a diameter of the sections, and AB and $\Gamma\Delta$ are drawn as ordinates [according to Proposition II.28]. Let $ENMH$ be drawn from E parallel to AB and $\Gamma\Delta$. Therefore EMH cuts ΘP each of the sections because there is no other meeting besides A, B, Γ, Δ . Then in one of the sections NM will be equal to EN , and in other section NE will be equal to NH [according to Definition 4], so that NM is equal to NH , but this is impossible ²⁹⁻³⁰.

[Proposition] 26

If the lines [of the sections] mentioned above some touch at one point, then they will not meet each other at more than two other points ³¹.

Let two of the above mentioned lines touch at the point A .

I say that they will not meet each other at more than two other points.

[Proof]. For let, if possible, them meet at B, Γ, Δ , and let the points of meeting be taken in succession with no point of meeting between them be left out. Let $B\Gamma$ be joined and continued, and from A let $A\Lambda$ be drawn touching the section. Thus $A\Lambda$ will touch both sections and meet ΓB . Let it meet it at Λ , and let it be that as $\Gamma\Lambda$ is to ΛB , so $\Gamma\Pi$ is to ΠB .

Let $\Delta\Pi$ be joined and continued. Thus it will meet the section and the straight lines drawn from the points of meeting to Λ will touch the section [according to Proposition IV.1]. Let it meet it at Θ and P , and let $\Theta\Lambda$ and ΛP be joined. These straight lines will touch the section. Therefore the straight line joining Δ to Λ will cut each of sections, and the earlier mentioned absurdity will occur. The section will not cut one another at more than two points.

If in an ellipse or the circumference of a circle ΓB is parallel to $A\Lambda$, the proof will be similar to that given above once $A\Theta$ is shown to be a diameter.

[Proposition] 27

If the lines [of the sections] mentioned above some touch one another at two points, they will not meet one another at another point ³².

Let two of lines mentioned above touch one another at two points

A and B. I say that they will not meet one another at another point.

[Proof]. For let, if possible, them meet also at Γ , and to start let Γ be outside of the points of contact A and B, and let straight lines be drawn from A and B touching the sections. Therefore they will touch both lines. Let them touch and be continued to Λ , as in the first diagram, and let $\Gamma\Lambda$ be drawn. Then it cuts each of the sections. Let it cut them at H and M, and let ANB be joined. Therefore in one of the sections as ΓN will be to NH, so $\Gamma\Lambda$ will be to ΛH , and in the other section as ΓN will be to NM, so [according to Proposition III.37] $\Gamma\Lambda$ will be to ΛM , but this is impossible.

[Proposition] 28

If ΓH is parallel to the straight lines touching the sections at A and B as in the ellipses in the second diagram³³, then joining AB we conclude that it is a diameter [according to Proposition II.27], so that each of ΓH and ΓM are bisected at N [according to Definition 4], but it is impossible. Therefore the lines [of the sections] do not meet one another at another point, but only at A and B

[Proposition] 29

Let Γ be between the points of contact, as in the third diagram³⁴. It is evident that the lines [of the sections] do not touch one another at Γ since it has been assumed that the lines [of the sections] touch at two points only. Indeed, let them cut one another [point] at Γ . Let $A\Lambda$ and ΛB be drawn from A and B touching the sections, let AB be joined and bisected at Z. Therefore the straight line drawn from Λ to Z [according to Proposition II.29] will be a diameter. The diameter will surely not pass through Γ for if it did pass through it, then the straight line drawn through Γ parallel to AB will touch each of the sections [according to Propositions II.5 and II.6], and this is impossible.

So from Γ let $\Gamma K H M$ be drawn parallel to AB, then in the one section ΓK will be equal to KH, and in the other section KM will be equal to $K\Gamma$, so that KM is equal to KH, but this is impossible.

Similarly if the straight lines touching the sections are parallel, the absurdity will be proved in the same way as above.

[Proposition] 30

*A parabola cannot touch a parabola at more points than one*³⁵.

[Proof]. For let, if possible, the parabolas AHB and AMB touch at A and B, and let $A\Lambda$ and ΛB be drawn touching the parabolas. They will, thus, touch both sections and will meet at Λ . Let AB be joined and bisected at Z, and let AZ be drawn.

Now since two lines AHB and AMB touch one another at A and B, [according to Propositions IV.27, IV.28, and IV.29] they will not meet each other at another point, so that ΛZ cuts each of sections. Let it cut them at H and M. In one section [according to Proposition I.35] ΛH will be equal to HZ, and in the other section ΛM

will be equal to MZ , but it is impossible. Therefore a parabola cannot touch a parabola at more points than one.

[Proposition] 31

A parabola falling outside of a hyperbola will not touch the hyperbola at two points ³⁶.

[Proof]. For let there be the parabola AHB and the hyperbola AMB , and, if possible, let them touch at A and B . Let the straight lines be drawn from A and B touching each of sections that touch at A and B , and let these straight lines meet at Λ . Let AB be joined and bisected at Z , and let ΛZ be joined.

Now since the sections AHB and AMB touch at A and B , they will not meet at another point, therefore ΛZ cuts the sections at one and then another point. Let it cut them at H and M and let ΛZ be continued. It will [according to Proposition II.29] fall on the center Δ of the hyperbola. According to the properties of the hyperbola as $Z\Delta$ is to ΔM , so $M\Delta$ is to $\Delta\Lambda$ and the remainders ZM to $M\Lambda$ [according to Proposition I.37]. Therefore ZM is greater than $M\Lambda$.

But according to the properties of the parabola [proved in Proposition I.35] ZH is equal to $H\Lambda$, but this is impossible.

[Proposition] 32

A parabola falling inside of an ellipse or the circumference of a circle will not touch the ellipse or the circumference of the circle at two points ³⁷.

[Proof]. For let there be the ellipse or the circumference of a circle AHB and the parabola AMB , and, if possible, let them touch at two points A and B , and let straight lines be drawn from A and B touching the sections and meeting at Λ , let AB be joined and bisected at Z , and let ΛZ be joined. ΛZ will cut each section at one point and then at another [point], as we said above. Let it cut them at H and M , and let ΛZ be continued to Δ , which is the center of the ellipse or of the circle. Therefore according to the properties of the ellipse and of the circle as $\Lambda\Delta$ is to ΔH , so ΔH is to ΔZ , and [according to Proposition I.37] that ratio is equal to the ratio of the remainders ΛH to HZ , and $\Lambda\Delta$ is greater than ΔH . Therefore ΛH is greater than HZ . But according to the properties of the parabola [proved in Proposition I.35] ΛM is equal to MZ , but this is impossible.

[Proposition] 33

A hyperbola will not touch a hyperbola with the same center at two points³⁸.

[Proof]. For let, if possible, the hyperbolas AHB and AMB with the same center Δ touch at A and B . Let $A\Lambda$ and ΛB be drawn from A and B touching the hyperbolas and meeting one another, and let $\Delta\Lambda$ be joined and continued. Moreover let AB be joined. Therefore ΔZ bisects AB at Z . Then ΔZ [according to Proposition IV.29] cuts the hyperbolas at H and M . According to the properties of the hyperbola AHB pl. $Z\Delta\Lambda$ will be equal to $\text{sq.}\Delta H$, and according to the properties of the hyperbola AMB pl. $Z\Delta\Lambda$ will be

equal to $\text{sq.}\Delta M$ [according to Proposition I.37]. Therefore $\text{sq.}M\Delta$ is equal to $\text{sq.}\Delta H$, but this is impossible.

[Proposition] 34

If an ellipse touches an ellipse or the circumference of a circle with the same center at two points, then the straight line joining the points of contact passes through falls on the center ³⁹.

[Proof] . For let the above mentioned lines touch one another at A and B. Let AB be joined, and let straight lines touching the sections be pass through A and B, and, if possible, meeting at Λ . Let AB be bisected at Z, and let ΛZ be joined. Therefore [according to Proposition II.29] ΛZ is a diameter of the sections. If possible, let the center be Δ . Therefore $\text{pl.}\Lambda\Delta Z$ will be equal to $\text{sq.}\Delta H$ according to the properties of one section, but to $\text{sq.}M\Delta$ according to the properties of other section, so that [according to Proposition I.37] $\text{sq.}H\Delta$ is equal to $\text{sq.}\Delta M$, but this is impossible. Therefore the straight lines from A and B touching the sections do not meet. Therefore they are parallel, and for the same reason AB is a diameter [according to Proposition II.27], so that it passes through the center, what was to prove ⁴⁰ .

[Proposition] 35

A section of a cone or the circumference of a circle will not meet a section of a cone or the circumference of a circle not having its convexity in the same direction at more than two points ⁴¹.

[Proof]. For let, if possible, a section of a cone or the circumference of a circle $AB\Gamma$ meet a section of a cone or the circumference of a circle $A\Delta BE\Gamma$ not having its convexity in the same direction at more points than two, A, B, Γ .

Since three points A, B, Γ have been taken on the line $AB\Gamma$, if AB and $B\Gamma$ are joined, they will contain an angle having concavity in the same direction as the line $AB\Gamma$. For the same reason $AB\Gamma$ contain an angle whose concavity is in the same direction as the line $A\Delta BE\Gamma$. Therefore the lines we have been speaking of have both their concave and convex parts in the same direction, but this is impossible.

[Proposition] 36

If a section of a cone or the circumference of a circle meets one of opposite hyperbolas at two points and the lines between the points of meeting have their concavity in the same direction, then the line drawn at the points of meeting will not meet the other opposite hyperbola ⁴².

Let there be the opposite hyperbolas Δ and $A\Gamma Z$, and let there be a section of a cone or the circumference of a circle ABZ meeting one of two opposite hyperbolas at two points A and Z, and let the sections ABZ and $A\Gamma Z$ have their concavity in the same direction.

I say that continued ABZ will not meet the section Δ .

[Proof]. For let AZ be joined. Since Δ and $A\Gamma Z$ are opposite hyperbolas and AZ cuts a hyperbola at two points, so continued it will not meet the opposite hyperbola Δ [according to Proposition II.33]. Neither therefore will the line ABZ meet the hyperbola Δ .

[Proposition] 37

If a section of a cone or the circumference of a circle meets one of the opposite hyperbolas it will not meet the remaining hyperbola at more points than two ⁴³.

Let there be the opposite hyperbolas A and B, and let a section of a cone or the circumference of a circle $AB\Gamma$ meet the hyperbola A, and let $AB\Gamma$ cut the opposite hyperbola B at B and Γ .

I say that it will not meet $B\Gamma$ at another point.

[Proof]. For let, if possible, it meet $B\Gamma$ at Δ . Therefore $B\Gamma\Delta$ meets the section $B\Gamma$ not having its concavity in the same direction at more points than two, but [according to Proposition IV.35] it is impossible.

This is will be shown similarly if the line $AB\Gamma$ touches the opposite hyperbola.

[Proposition] 38

A section of a cone or the circumference of a circle will not meet opposite hyperbolas at more points than four ⁴⁴.

This is evident from the fact that meeting one of the opposite hyperbolas it [according to Proposition IV.37] cannot meet the remaining hyperbola at more than two points.

[Proposition] 39

If a section of a cone or the circumference of a circle touches one of the opposite hyperbolas in the concave part of the latter it will not meet the other opposite hyperbola ⁴⁵.

Let there be the opposite hyperbolas A and B, and let $\Gamma A\Delta$ touch the hyperbola A [from the direction of its concavity].

I say that $\Gamma A\Delta$ will not meet the hyperbola B.

[Proof]. For let EAZ be drawn from A touching the hyperbola A. Then it touches each of the sections [A and $\Gamma A\Delta$] at A, hence [according to Proposition II.30] it will not meet [the hyperbola] B, so that neither will $\Gamma A\Delta$ meet B.

[Proposition] 40

If a section of a cone or the circumference of a circle touches each of two opposite hyperbolas at one point, it will not meet the opposite hyperbolas at other point ⁴⁶.

Let there be the opposite hyperbolas A and B, and let a section of a cone or the circumference of a circle touch each of the hyperbolas A and B at

the points A and B.

I say that the line $AB\Gamma$ will not meet the hyperbolas A and B at another point.

[Proof]. Indeed since the line $AB\Gamma$ touches the hyperbola A and meets [the hyperbola] B at one point, therefore it will not touch A in the direction of its concavity. Similarly it will be shown that neither will it touch B in the direction of its concavity. Let $A\Delta$ and BE be drawn touching the hyperbolas A and B, then they will touch the line $AB\Gamma$. For, if possible, let one of them cut the line [of the section] and let it be AZ . Therefore between AZ touching the hyperbola A, and the hyperbola A, a straight line AH is situated, but this is impossible. Therefore it touches $AB\Gamma$, and because of this it is evident that $AB\Gamma$ does not meet the opposite hyperbolas at another point.

[Proposition] 41

If a hyperbola meets one of the opposite hyperbolas at two points having its convexity in the opposite direction to the concavity of the touching hyperbola, then the opposite hyperbola of the mentioned hyperbola will not meet the other opposite hyperbola ⁴⁷ .

Let there be the opposite hyperbolas $AB\Delta$ and Z, let the hyperbola $AB\Gamma$ meet $AB\Delta$ at A and B, the former [of them] has its convexity in the opposite direction to the concavity of the latter, and let E be the opposite hyperbola of $AB\Gamma$.

I say that E will not meet Z.

[Proof]. For let AB be joined and continued to H. Since indeed the straight line ABH cuts the hyperbola $AB\Delta$ and continued it falls outside of each section, it [according to Proposition II.33] will not meet the hyperbola Z. Similarly because ABH cuts the hyperbola $AB\Gamma$, it will not meet the opposite hyperbola E, therefore neither will E meet Z.

[Proposition] 42

If a hyperbola meets each of two opposite hyperbolas, its opposite hyperbola will meet neither of the opposite hyperbolas at two points ⁴⁸ .

Let there be the opposite hyperbolas A and B, and let the hyperbola $A\Gamma B$ meets each of the opposite hyperbolas A and B.

I say that the opposite hyperbola of $A\Gamma B$ will not meet the hyperbolas A and B at two points.

[Proof]. For let, if possible, it meet one of the opposite hyperbola at Δ and E, and let ΔE be joined and continued. Because of the hyperbola ΔE the straight line ΔE [according to Proposition II.33] will not meet the hyperbola AB, and on the other hand because of the section $AE\Delta$ [the straight line] ΔE will not meet the hyperbola B since it passed through the three places [according to Proposition II.33], but this is impossible. Similarly it will be shown that $A\Gamma B$ will not meet B at two points.

For the same reasons neither will it touch either of the opposite hyperbolas for drawing ΘE touching it will touch each of the hyperbolas, so that, because of the hyperbola ΔE it will not meet the hyperbola $A\Gamma$, whereas because of the hyperbola AE

will it not meet the hyperbola B, so that neither will $A\Gamma$ meet B, but this is contrary to what was assumed.

[Proposition] 43

If a hyperbola cuts each of two opposite hyperbola at two points having its convexity in the opposite direction to each of them, the opposite hyperbola of the mentioned hyperbola will meet neither of the mentioned opposite hyperbolas ⁴⁹.

Let there be the opposite hyperbolas A and B, and let the hyperbola $\Gamma A B \Delta$ cut each of the hyperbolas A and B at two points containing convexities in the opposite directions.

I say that the opposite hyperbola EZ [of $\Gamma A B \Delta$] meets neither of the hyperbolas A and B.

[Proof]. For let, if possible, it meet the hyperbola A at E, and let ΓA and ΔB be joined and continued, then these straight lines will meet one another [according to Proposition II.25]. Let them meet at Θ situated in the angle between the asymptotes of the hyperbola $\Gamma A B \Delta$ [according to Proposition II.25]. And EZ is the opposite hyperbola of $\Gamma A B \Delta$. Therefore the straight line joining E to Θ will fall in the angle $A\Theta B$. Again since $\Gamma A E$ is a hyperbola and $\Gamma A \Theta$ and ΘE meet, and the points of meeting Γ and A do not contain E, the point Θ will be between the asymptotes of the hyperbola $\Gamma A E$. And $B \Delta$ is the opposite hyperbola of $\Gamma A E$. Therefore the straight line from B to Θ falls inside of the angle $\Gamma \Theta E$, but this is impossible for it also fall in the angle $A\Theta B$. Therefore EZ will not meet one of the opposite hyperbola A and B.

[Proposition] 44

If a hyperbola cuts one of two opposite hyperbolas at four points, the opposite hyperbola of the hyperbola will not meet the other of the two opposite hyperbolas ⁵⁰.

Let there be the opposite hyperbolas $A B \Gamma \Delta$ and E, and let a hyperbola cut $A B \Gamma \Delta$ at four points A, B, Γ , Δ , and let its opposite hyperbola be K. I say that K will not meet E.

[Proof]. For let, if possible, it meet it at K. Let AB and $\Gamma \Delta$ be joined and continued, then they will meet one another. Let them meet at Λ , and let as $A \Pi$ be to ΠB , so $A \Lambda$ be to ΛB , and let as ΔP be to $P \Gamma$, so $\Delta \Lambda$ be to $\Lambda \Gamma$.

Therefore the straight line through Π and P will meet the hyperbolas on each side, and the straight lines from L to the points of meeting will touch the hyperbolas [according to Proposition IV.9]. Let $K \Lambda$ be joined and continued. It will cut the angle $B \Lambda \Gamma$ and the hyperbolas at one and then another point. Let it cut them at Z and M [according to the properties of the opposite hyperbolas $A B \Gamma \Delta$ and E as $N K$ is to $K \Lambda$, so $N M$ is to $M \Lambda$, but this is impossible. Therefore E and K will not meet one another.

[Proposition] 45

If a hyperbola meets one of two opposite hyperbolas at two points

*having its concavity in the same direction as the hyperbola, and it meets the other of two opposite hyperbolas at one point, then the opposite hyperbola of the mentioned hyperbolas will meet neither of the opposite hyperbolas*⁵¹.

Let there be the opposite hyperbolas AB and Γ , and let the hyperbola $A\Gamma B$ meet AB at the points A and B and let it meet the hyperbola Γ at one point, and let Δ be the opposite hyperbola of $A\Gamma B$.

I say that Δ will meet neither of the hyperbola AB and Γ .

[Proof]. For let $A\Gamma$ and $B\Gamma$ be joined and continued. Therefore $A\Gamma$ and $B\Gamma$ will not meet the hyperbola Δ [according to Proposition II.33]. Neither will they meet the hyperbola Γ at another point besides Γ for if they meet the hyperbola Γ at another point they will not meet the opposite hyperbola AB [according to Proposition II.33], where it is assumed that they do meet.

Therefore the straight lines $A\Gamma$ and $B\Gamma$ meet the hyperbola Γ at one point Γ , and they do not meet Δ at all. Therefore Δ will be in the angle $E\Gamma Z$, so that the hyperbola Δ will not meet AB and Γ .

[Proposition] 46

*If a hyperbola meets one of two opposite hyperbolas at three points, the opposite hyperbola of the hyperbola will not meet the other opposite hyperbola at more than one point*⁵².

Let there be the opposite hyperbolas $AB\Gamma$ and ΔEZ , and let the hyperbola $AMB\Gamma$ meet $AB\Gamma$ at three points A , B , and let ΔK be opposite hyperbola of $AM\Gamma$.

I say that ΔK will not meet ΔEZ at more point than one.

[Proof]. For let, if possible, them meet at Δ and E , and let AB and ΔE be joined. Now they will either be parallel or not.

To start let them be parallel, and let AB and ΔE be bisected at H and Θ , and let $H\Theta$ be joined, therefore $H\Theta$ is a diameter for all these hyperbolas [according to Proposition II.36], and AB and ΔE are drawn as ordinates. Let $\Gamma N\Theta$ be drawn from Γ parallel to AB , then it will be drawn as an ordinate to the diameter, and it will cut the hyperbolas, one and then other for if it were to cut them at the same point, the hyperbolas would no longer meet at three points, but as four. In the hyperbola AMB then ΓN will be equal to $N\Xi$, and in $A\Delta B$ then ΓN will be equal to NO . And therefore ON is equal to $N\Xi$, but this is impossible.

So let straight lines AB and ΔE not be parallel, but be continued. Let them meet at Π . Let ΓO be drawn parallel to $A\Pi$ and let it meet continued $\Delta\Pi$ at P . And let AB and ΔE be bisected at H and Θ , through H and Θ let diameters $H\Sigma I$ and $\Theta\Lambda M$ be drawn, and from I , Λ , and M let IYT , MY , and ΛT be drawn touching the hyperbola, then IT will be parallel to $\Delta\Pi$, and ΛT and MY will be parallel to $A\Pi$ and OP [according to Proposition II.5]. Since as $\text{sq.}MY$ is to $\text{sq.}YI$, so $\text{pl.}A\Pi B$ is to $\text{pl.}\Delta\Pi E$ [according to Proposition III.19], but as $\text{pl.}A\Pi B$ is to $\text{pl.}\Delta\Pi E$, so $\text{sq.}\Lambda T$ is to $\text{sq.}TI$, and therefore as $\text{sq.}MY$ is to $\text{sq.}YI$, so $\text{sq.}\Lambda T$ is to $\text{sq.}TI$.

For the same reasons as sq. MY is to sq. YI , so $\text{pl. } \Xi\text{P}\Gamma$ is to $\text{pl. } \Delta\text{PE}$, as $\text{sq. } \Lambda\text{T}$ is to sq. TI , so $\text{pl. } \text{OP}\Gamma$ is to $\text{pl. } \Delta\text{PE}$. Therefore $\text{pl. } \text{OP}\Gamma$ is equal to $\text{pl. } \Xi\text{P}\Gamma$, but this is impossible.

[Proposition] 47

If a hyperbola touches one of two opposite hyperbolas, and it cuts the other at two points, then the opposite hyperbola of the hyperbola will meet neither of the opposite hyperbolas. ⁵³

Let there be the opposite hyperbolas $\text{AB}\Gamma$ and Δ , and some hyperbola $\text{AB}\Delta$ cut $\text{AB}\Gamma$ at A and B, and touch the hyperbola Δ at the point Δ , and let ΓE be the opposite hyperbola of $\text{AB}\Delta$.

I say that ΓE meets neither of the opposite hyperbolas $\text{AB}\Gamma$ and Δ .

[Proof]. For let, if possible, let ΓE meet $\text{AB}\Gamma$ at Γ , and let AB be joined, and let a straight line be drawn through Δ touching the hyperbola $\text{AB}\Delta$ and meeting AB at Z .

Therefore Z [according to Proposition II.25] will be inside of the angle between the asymptotes of the hyperbola $\text{AB}\Delta$. And ΓE is the opposite hyperbola of $\text{AB}\Delta$. Therefore the straight line from Γ to Z falls inside of the angle $\text{BZ}\Delta$. Again since $\text{AB}\Gamma$ is a hyperbola, and AB and ΓZ meet, and the points of meeting A and B do not contain Γ , the point Z is between the asymptotes of the hyperbola $\text{AB}\Gamma$. And Δ is the opposite hyperbola of $\text{AB}\Gamma$. Therefore the straight line from ΔZ falls inside of the angle $\text{AZ}\Gamma$, but it is impossible for it fell in the angle $\text{BZ}\Delta$. Therefore ΓE does not meet one of the opposite hyperbolas $\text{AB}\Gamma$ and Δ .

[Proposition] 48

If a hyperbola touches one of two opposite hyperbolas at one point, and it meets it at two points, then the opposite hyperbola of the hyperbola will not meet the other opposite hyperbola ⁵⁴.

Let there be the opposite hyperbolas $\text{AB}\Gamma$ and Δ , and let some hyperbola $\text{AH}\Gamma$ touch $\text{AB}\Gamma$ at A, and let it meet $\text{AB}\Gamma$ at B and Γ , and let E be the opposite hyperbola of $\text{AH}\Gamma$.

I say that E will not meet Δ .

[Proof]. For let, if possible, E meet it at Δ , let $\text{B}\Gamma$ be joined and continued to Z , and let AZ be drawn from A touching the hyperbola. As in the earlier proof it will be shown that Z is inside of the angle between the asymptotes [according to Proposition II.25]. Moreover AZ will touch both hyperbolas, and continued ΔZ will cut the sections at H and K between A and B. Let as $\Gamma\Lambda$ is to ΛB , so ΓZ is to ZB , and let $\Lambda\Lambda$ be joined and continued, it will cut the hyperbolas, one and then other [according to Proposition IV.1]. Let it cut them at N and M. Therefore the straight lines from Z to N and M will touch the hyperbolas [according to Proposition IV.1], and as in the earlier proof [according to the Proposition III.37] according to the properties of the one hyperbola as ΞK is to KZ , so $\Xi\Delta$ is to ΔZ , and according to the properties of the other

hyperbola as ΞH is to HZ , so $\Xi \Delta$ is to ΔZ , but this is impossible. Therefore it does not meet the opposite hyperbola.

[Proposition] 49

If a hyperbola touching one of two opposite hyperbolas meets the same hyperbola at another point, then the opposite hyperbola of the hyperbola will not meet the other opposite hyperbola at more points than one ⁵⁵.

Let there be the opposite hyperbolas $AB\Gamma$ and EZH , and let some hyperbola $\Delta A\Gamma$ touch $AB\Gamma$ at A , and let it cut $AB\Gamma$ at Γ , and let $EZ\Theta$ be the opposite hyperbola of $\Delta A\Gamma$.

I say that it will not meet the other opposite hyperbola at more points than one.

[Proof]. For let, if possible, let it meet it at two points E and Z , and let EZ be joined and through A let AK be drawn touching the hyperbolas. Now EZ and AK will be parallel or not parallel.

To start let them be parallel, and let the diameter bisecting EZ be drawn, therefore it will pass through A and it will be the diameter of two conjugate hyperbolas [according to Proposition II.34]. Let $\Gamma\Lambda\Delta B$ be drawn through Γ parallel to AK and EZ . Therefore it will cut the hyperbolas at one and then at another point. Then in the one hyperbola $\Gamma\Lambda$ will be equal to $\Lambda\Delta$, and in the remaining hyperbola $\Gamma\Lambda$ will be equal to LB , but this is impossible.

So, let AK and EZ not be parallel, let them meet at K , and let $\Gamma\Delta$ drawn parallel to AK meet EZ at N . Let AM bisecting EZ cut the hyperbolas at Ξ and O , and let $\Xi\Pi$ and OP be drawn from Ξ and O touching the hyperbolas. Therefore as $\text{sq.}\Pi\Xi$ is to $\text{sq.}\Xi E$, so $\text{sq.}\Pi P$ is to $\text{sq.}\Xi O$, and for this reason as $\text{pl.}\Delta N\Gamma$ is to $\text{pl.}\Xi N Z$, and as $\text{pl.}\Pi N\Gamma$ is to $\text{pl.}\Xi N Z$. Therefore $\text{pl.}\Delta N\Gamma$ is equal to $\text{pl.}\Pi N\Gamma$, but this is impossible.

[Proposition] 50

If a hyperbola touches one of two opposite hyperbolas at one point, the opposite hyperbola of the hyperbola will not meet other opposite hyperbola at more points than two ⁵⁶.

Let there be the opposite hyperbolas AB and $E\Delta H$, and let a hyperbola $A\Gamma$ touch AB at A , then let $E\Delta Z$ be the opposite hyperbola of $A\Gamma$.

I say that $E\Delta Z$ will not meet $E\Delta H$ at more points than two.

[Proof]. For let, if possible, $E\Delta Z$ meet $E\Delta H$ at three points Δ , E , and Θ , let AK be drawn touching hyperbolas AB and $A\Gamma$, let ΔE be joined and continued, and, start, let AK and ΔE be parallel. Let ΔE be bisected at Λ , and let $A\Lambda$ be joined. Then $A\Lambda$ be a diameter for two conjugate hyperbolas [according to Proposition II.34], and will cut the hyperbola between Δ and E at M and Z . Let ΘZH be drawn from Θ parallel to ΔE . Then in the one section $\Theta\Xi$ will be equal to ΞZ , and in the other section $\Theta\Xi$ will be equal to ΞH , so that also ΞZ is equal to ΞH , but this is impossible.

So let AK and ΔE not be parallel, but let them meet at K , and let the remaining constructions be the same. Let AK be continued and let it meet $Z\Theta$ at P . As

before we will show that [according to Proposition III.19] in the hyperbola $Z\Delta E$ as pl.ZP Θ is to sq.PA, so pl. ΔKE is to sq.AK, and in the hyperbola $H\Delta E$ as pl.HP Θ is to sq.PA, so pl. ΔKE is to sq.AK. Therefore pl.HP Θ is equal to pl.ZP Θ , but this is impossible. Therefore $E\Delta Z$ does not meet $E\Delta H$ at more points than two.

[Proposition] 51

If a hyperbola touches two opposite hyperbolas, the opposite hyperbola of the hyperbola will meet neither of the opposite hyperbolas ⁵⁷.

Let there be the opposite hyperbolas A and B , and let the hyperbola AB touch each of them at the points A and B , and let the opposite hyperbola of AB be E . I say that E will meet neither of the hyperbolas A and B .

[Proof]. For let, if possible, it meet A at Δ , and let straight lines be drawn from A and B touching the hyperbolas, they will meet one another hyperbola in the angle between the asymptotes of the hyperbola AB [according to Proposition II.25]. Let them meet at Γ , and let $\Gamma\Delta$ be joined. Therefore $\Gamma\Delta$ will be in the place between $A\Gamma$ and ΓB . But it is between $B\Gamma$ and ΓZ , it is impossible. Therefore E does not meet A and B .

[Proposition] 52

If each of two opposite hyperbolas touch each of two opposite hyperbolas at one point, each having its concavity in the same direction, then they will not meet at another point ⁵⁸.

Let the opposite hyperbolas touch one another at A and Δ .

I say that they will not meet at another point.

[Proof]. For let, if possible, them meet at E . Since, indeed, a hyperbola touching one of the opposite hyperbolas meets at E , therefore the hyperbola AB will not meet the hyperbola $A\Gamma$ at more points than one [according to Proposition IV.49]. Let $A\Theta$ and $\Theta\Delta$ be drawn from A and Δ touching the hyperbolas, let $A\Delta$ be joined, let $EB\Gamma$ be drawn through E parallel to $A\Delta$, and let the second diameter $\Theta K\Lambda$ of the opposite hyperbolas be drawn from Θ [according to Proposition II.38]. Then it will bisect $A\Delta$ at K . And therefore EB and $E\Gamma$ will be bisected at Λ [according to Proposition II.39]. Therefore $B\Lambda$ is equal to $\Lambda\Gamma$, but it is impossible. Therefore the hyperbolas will not meet at another point.

[Proposition] 53

If a hyperbola touches one of two opposite hyperbolas at two points, the opposite hyperbola of the hyperbola will not meet other opposite hyperbola ⁵⁹.

Let there be the opposite hyperbolas $A\Delta B$ and E , and let the hyperbola $A\Gamma$ touch $A\Delta B$ at two points A and B , and let Z be the opposite hyperbola of $A\Gamma$.

I say that Z will not meet E .

[Proof]. For let, if possible, it meet it at E , and let AH and HB be drawn from A and B touching the hyperbolas, let AB and EH be joined, and let EH be continued, it will cut the hyperbolas at one and then at another point, let it be as $EH\Gamma\Delta\Theta$. Since AH

and HB indeed touch the hyperbola, and AB joins the points of contact in one of the conjugate hyperbolas as $O\Delta$ is to ΔH , so ΘE is to EH , and in other hyperbola as $\Theta\Gamma$ is to ΓH , so ΘE is to EH , but it is impossible. Therefore the hyperbola Z does not meet the hyperbola E .

[Proposition] 54

If a hyperbola touches one of two opposite hyperbolas with the convexities in the opposite directions, then the opposite hyperbola of the hyperbola will not meet other opposite hyperbola ⁶⁰.

Let there be the opposite hyperbolas A and B , and some hyperbola $A\Delta$ touch the hyperbola A at the point A , and let the opposite hyperbola of $A\Delta$ be Z . I say that Z will not meet B .

[Proof]. For let $A\Gamma$ be drawn from A touching the hyperbolas, therefore because of the properties of the hyperbola $A\Delta$ [the straight line] $A\Gamma$ will not meet Z , and because of the properties of the hyperbola A [according to Proposition II.33] it will not meet B , so that $A\Gamma$ falls between the hyperbolas B and Z . Then it is evident that B will not meet Z .

[Proposition] 55

Opposite hyperbolas will not meet opposite hyperbolas at more points than four ⁶¹.

Let there be one pair of opposite hyperbolas AB and $\Gamma\Delta$, and let another pair of opposite hyperbolas be $AB\Gamma\Delta$ and EZ , and, to start let $AB\Gamma\Delta$ cut each of AB and $\Gamma\Delta$ at four points A , B , Γ , and Δ containing convexities in opposite directions, as in the first diagram. Therefore the opposite hyperbola of $AB\Gamma\Delta$, that is EZ , will not meet AB and $\Gamma\Delta$ [according to Proposition IV.43].

But let $AB\Gamma\Delta$ cut AB at A and B and Γ at one point Γ , as in the second diagram. Therefore EZ does not meet the hyperbola Γ [according to Proposition IV.41]. If EZ meets AB , it will meet it at one point only for if it meets it at two points, its opposite hyperbola $AB\Gamma$ will not meet other opposite hyperbola Γ [according to Proposition IV.43]. But it has been assumed that it meets it at one point Γ .

If, as in the third diagram, $AB\Gamma$ cuts ABE at two points A and B , and EZ meets ABE at one point, EZ will not meet the hyperbola Δ [according to Proposition IV.41], where as meeting ABE it will not meet ABE at more points than two.

If, as in the fourth diagram, $AB\Gamma\Delta$ cuts each of two opposite hyperbolas at one point, EZ will meet neither at two points [according to Proposition IV.42]. [So that according to already said and its converse, $AB\Gamma\Delta$ and ΓZ will not meet the opposite hyperbolas BE and EZ at more points than four] ⁶². If the hyperbolas have their concavities in the same direction and one cuts other at four points A , B , Γ , and Δ , has in the fifth diagram, EZ will not meet other opposite hyperbola [according to Proposition IV.44]. Of course, EZ will not meet AB for again AB will not meet the opposite hyperbo-

las $AB\Gamma\Delta$ and EZ at more points than four [according to Proposition IV.38], neither will $\Gamma\Delta$ meet EZ .

If, as in the sixth diagram, $AB\Gamma\Delta$ meets other hyperbola at three points, EZ will meet other hyperbola at one point only [according to Proposition IV.46].

And we will say the same as before for the remaining cases.

So, since what was proposed is clear in all possible configurations, opposite hyperbolas will not meet opposite hyperbolas at more points than four.

[Proposition] 56

If opposite hyperbolas touch opposite hyperbolas at one point, they will not meet at more than two other points ⁶³.

Let there be the opposite hyperbolas AB and $\Gamma\Delta$ and others Δ and EZ , let $B\Gamma\Delta$ touch AB at B , let their convexities in opposite directions, and, first, let $B\Gamma\Delta$ meet $\Gamma\Delta$ at two points Γ and Δ , as in the first diagram.

Indeed since $B\Gamma\Delta$ cuts $\Gamma\Delta$ at two points having their convexities in opposite directions, EZ will not meet AB [according to Proposition IV.41]. Again since $B\Gamma\Delta$ touches AB at B , and their convexities are in opposite directions, EZ will not meet $\Gamma\Delta$ [according to Proposition IV.54]. Therefore EZ will not meet either the hyperbolas AB and $\Gamma\Delta$, therefore these hyperbolas will meet at two points Γ and Δ only.

But let $B\Gamma$ cut $\Gamma\Delta$ at one point Γ , as in the second diagram. Therefore EZ will not meet $\Gamma\Delta$ [according to Proposition IV.54], whereas it will meet AB at one point only for if EZ meets AB at two points, $B\Gamma$ will not meet $\Gamma\Delta$ [according to Proposition IV.41]. But it was assumed that they meet at one point.

If $B\Gamma$ does not meet the hyperbola Δ , as in the third diagram, then according to what has been said above, EZ will not meet Δ [according to Proposition IV.54], whereas EZ will not meet AB at more points than two [according to Proposition IV.37].

If the hyperbolas have their concavities in the same direction, the same proof will applied.

So, from that proof, what was proposed is clear in all possible configurations.

[Proposition] 57

If opposite hyperbolas touch opposite hyperbolas at two points, they will not meet at another point ⁶⁴.

Let there be the opposite hyperbolas AB and $\Gamma\Delta$, and others $A\Gamma$ and EZ , and first, let them touch at A and Γ , as in the first diagram.

Indeed since $A\Gamma$ touches each of the hyperbolas AB and $\Gamma\Delta$ at A and Γ , therefore EZ will meet neither on the hyperbolas AB and $\Gamma\Delta$ [according to Proposition IV.51].

So, let them touch as in the second diagram. It will be proved similarly that $\Gamma\Delta$ will not meet EZ [according to Proposition IV.53].

So, let ΓA touch AB at A and let Δ touch EZ at Z , as in the third

diagram. Indeed, since $A\Gamma$ touches AB having their convexities in opposite directions, EZ will not meet AB . Again, since $Z\Delta$ touches EZ , ΓA will not meet ΔZ .

If $A\Gamma$ touches AB at A , and $E\Gamma$ touches $\Gamma\Delta$ at Γ , and their concavities are in the same direction, as in the fourth diagram, they will not meet at another point [according to Proposition IV.52]. EZ will not even meet AB .

So, from the proposed proof it is clear in all possible configurations ⁶⁵ .