MATH 311M

POSSIBLE TOPICS FOR A WRITING PROJECT

1. Divisibility tricks: divisibility by 2, 3, 5, 7, 9, 11, and 13. Many of these tricks are based on the representation of an integer in different bases. In the decimal (base 10) system a number a is written as

$$a = (a)_{10} = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_1 10^1 + a_0,$$

where $a_0, a_1, \ldots, a_n$ are digits, i.e. integers $0 \leq a_i \leq 9$. Similarly, for an integer base b the same number a will be written as

$$a = (a)_b = c_k b^k + c_{k-1} b^{k-1} + \cdots + c_1 b^1 + c_0,$$

with digits $c_0, c_1, \ldots, c_k$, $0 \leq c_i \leq b - 1$.
(a) Prove that if b is an integer, then an integer a

(1) is divisible by any divisor p of $(b-1)$ if and only if the sum of digits in its expansion $(a)_b$ is divisible by p.
(2) is divisible by any divisor q of $(b+1)$ if and only if the alternating sum of digits in its expansion $(a)_b$ is divisible by q.

(b) Use (a) for $b = 10$, 100, and 1000 to derive the divisibility by 3, 7, 9, 11, and 13 tricks.

2. The Locker Problem A locker room has n lockers numbered 1 to n, and all are locked. n attendants P_1, P_2,..., P_n file through the room in order. The first attendant unlocks all lockers, the second attendant locks every second locker, and so on, i.e. the attendant P_k changes the status of every locker whose number is divisible by k (unlocks it if it was locked or locks it if it was unlocked). Which lockers will be locked and which will be unlocked after all n attendants have passed through the room? What is the situation if each attendant performs the same operation, but they file through in some other order?

3. Infinitude of primes, gaps, and primes in arithmetic progressions. Recall the Euclid's proof of the infinitude of prime numbers.

(1) Prove that the sequence of prime numbers has infinitely large gaps, for example, find a thousand of consecutive numbers which contains no prime number.
(2) Prove that there are infinitely many primes of the form $4n + 3$.
(3) Prove that there are infinitely many primes of the form $6n + 5$.
(4) Prove that there are infinitely many primes of the form $3n + 1$.
(5) * Try the same for numbers of the form $4n+1$, using $(2p_1 \ldots p_n)^2+1$.
(6) * Here is another proof of the infinitude of primes, due to G. Polya: Show that $2^{2^n}+1$ and $2^{2^m}+1$ are relatively prime if $n \neq m$. Assume that p is a prime divisor of $2^{2^n} + 1$ and prove that the order of 2 in $\mathbb{Z}_p^\times$ is equal to 2^{2^n} (first show that it is a power of 2). Use the above result to show that there are at least n primes less than $2^{2^n} + 1$ for each n, hence there are infinitely many primes.

4. Fibonacci numbers. The Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

is defined recursively as follows: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. Find a general formula using linear algebra.

Start with a matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Taking A^2, A^3 and so on, notice the following pattern:

$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \; A^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, \ldots$$

(1) Prove by induction that $A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$.

(2) Prove that the matrix A has two real eigenvalues λ and λ^{-1}, and find them.

(3) Find the matrix C which brings A to the diagonal form $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ so we have $A = C^{-1}\Lambda C$, and hence $A^n = C^{-1}\Lambda^n C$.

(4) Obtain the formula for F_n.

The eigenvalue $\lambda = \frac{1+\sqrt{5}}{2} = 1.618033...$ of the matrix A is called the *golden ratio*. It can be defined geometrically as the length of the longer side of a rectangle whose second side is of length 1 and such that if one cuts off a unit square of this rectangle, the obtained rectangle will be similar to the original one. The rectangle with sides 1 and λ represents a rectangle of "perfect shape" (see Fig. 6).

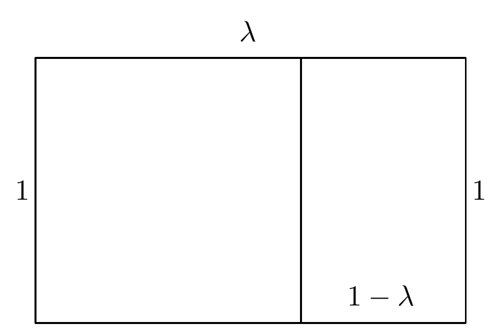

Figure 6

(5) Prove that $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$, the eigenvalue of the matrix A above.

(6) Prove that

$$\frac{F_n}{F_{n-1}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots + \cfrac{1}{1}}}},$$

a finite continued fraction of n terms. Deduce that $\frac{1+\sqrt{5}}{2}$ can be expressed as an infinite continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots + \cfrac{1}{\cdots}}}}.$$

5. Coloring of the plane.

(1) Points in the plane are each colored with one of three colors: red, green, or blue. Prove that, for a given distance d, there always exist two points of the same color at the distance d from each other.

(2) Points in the plane are each colored with one of two colors: red or blue. Prove that there is a rectangle whose vertices are of the same color.

6. Periodic infinite decimal fractions. When we transform usual fractions into decimals, we encounter the following typical examples:

$$(I) \quad \frac{1}{5} = 0.2, \qquad \frac{3}{40} = 0.075;$$
$$(II) \quad \frac{4}{9} = 0.4444\ldots, \qquad \frac{1}{7} = 0.142857142857\ldots;$$
$$(III) \quad \frac{1}{6} = 0.1666\ldots, \qquad \frac{7}{30} = 0.2333\ldots$$

We assume that in the fraction $\frac{a}{b}$, $(a, b) = 1$.

(1) Prove that any fraction $\frac{a}{b}$ transforms into an infinite, eventually periodic infinite decimal fraction, and the period contains not more than $b - 1$ digits.

In the following we assume that $(b, 10) = 1$.

(2) Prove that the period of $\frac{a}{b}$ contains not more than $\varphi(b)$ digits, where $\varphi(b)$ is the number of numbers between 1 and b relatively prime to b.

(3) Prove that the infinite decimal fraction $\frac{a}{b}$ is purely periodic, i.e. its period begins right after the decimal point.

(4) Prove that the length of the period of the infinite decimal fraction $\frac{a}{b}$ is the smallest number λ such that $10^\lambda - 1$ is divisible by b.

7. Symmetry groups of the regular solids. A *regular solid* is a 3-dimensional polyhedron in which each face is a regular polygon. Any two faces as well as any two vertices can be matched by an isometry (a rigid motion) of the 3-dimensional space. It is convenient to use a symbol $\{p, q\}$ for a regular solid whose faces are regular p-gons with q of them situated around each vertex.

There are FIVE regular solids:
TETRAHEDRON $\{3, 3\}$,
CUBE $\{4, 3\}$,
OCTAHEDRON $\{3, 4\}$,
DODECAHEDRON $\{5, 3\}$,
ICOSAHEDRON $\{3, 5\}$

Any regular solid may be inscribed in a sphere, and then any symmetry of any regular solid will leave the center of the sphere fixed and will transform the surface of the sphere onto itself. We call a *rotational symmetry* of a regular solid any rotation of the sphere (with respect to an axis passing through its center) mapping the regular solid into inself. A *symmetry* of a regular solid is either a rotational symmetry or a reflection with respect to a plane (also passing through the center of the sphere), mapping the regular solid onto itself.

Prove that

(1) The group of rotational symmetries of a tetrahedron is $A(4)$.
(2) The full group of symmetries of a tetrahedron is $S(4)$.
(3) The group of rotational symmetries of a cube or of an octahedron is $S(4)$.
(4) The group of rotational symmetries of an icosahedron or of a dodecahedron is $A(5)$.