2. Mathematical induction.

Notations: $\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$.

Let a be an integer, and $P(n)$ be an assertion involving an integer $n \geq a$. If

(a) $P(a)$ is true;
(b) for any $k \geq a$, $P(k)$ true implies $P(k+1)$ true,

then $P(n)$is true for every integer $n \geq a$.

Often $a = 1$ or $a = 0$.

Proof by induction:

(1) *Base case*: show that $P(a)$ holds.
(2) Assume that $P(k)$ is true (induction hypothesis), and prove that $P(k+1)$ is true.

Problems solved by mathematical induction come in two varieties: (i) when an answer is given, (ii) when you have to guess the answer first (usually, by considering small values of n and noticing a pattern, and then prove it by induction.

Problems on induction:

(1) Prove by induction that $1 + 2 + \cdots + n = \frac{(n+1)n}{2}$.
(2) Prove by induction that the sum of the arithmetic progression $\sum_{i-1}^{i=n} a_i = \frac{(a_1 + a_n)n}{2}$.
(3) Sum of the geometric progression

$$T_n = a + aq + aq^2 + \ldots aq^{n-1} = a\frac{q^n - 1}{q - 1}.$$

Give a direct proof and a proof by induction.

(4) Prove the divergence of the harmonic series by showing that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} > \frac{n}{2}.$$

$P(1)$: $1 > \frac{1}{2}$.
$P(k) \Rightarrow P(k+1)$:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{k+1}} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k}$$
$$+ (\frac{1}{2^k + 1} + \cdots + \frac{1}{2^{k+1}}) > \frac{k}{2} + \frac{2^k}{2^{k+1}}$$
$$= \frac{k}{2} + \frac{1}{2} = \frac{k+1}{2}.$$

(5) Discover that the sum of cubes is a square:

$$1^3 + 2^3 + \ldots n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

Find an answer by looking for a pattern, use 1. above.

(6) Binomial theorem (Theorem 1.1.1 in the book).

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \dots \binom{+}{n}ny^n,$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is a binomial coefficient.

$$n! = 1 \cdot 2 \cdot \dots \cdot n, \; 0! = 1.$$

Pascal triangle.

$$\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i}.$$

(7) The plane is divided into regions by drawing a finite number of straight lines. Show that it is possible to color each of these regions either red or blue in such a way that no two ajacent regions have the same color.

(8) Let P_n is the number of regions formed when n lines are drawn in the planein such a way that no three pass through the same point, and no two are parallel. Show that $P_{n+1} = P_n + (n+1)$, find a formula for P_n, and prove it by induction.

(9) *Where is a mistake?* All people are of the same height.

Strong induction. : If

(a) $P(a)$ holds;

(b) for each $k \geq a$, $P(a), P(a+1), \dots, P(k)$ true implies $P(k+1)$ true,

then $P(n)$ is true for every integer $n \geq a$.

Problems on strong induction:

(1) Prove that any integer $n \geq 2$ can be written as a product of prime numbers.

*Strong induction o*n n. The base case $a = 2$, since 2 is a prime, done. Assume the statement is true for $n = 2, 3, \dots, k$, and we prove it for $n = k+1$. If $k+1$ is prime, we are done. Assume it is not prime, then it can be written as a product $k+1 = k_1k_2$, where $k_1 < k+1$ and $k_2 < k+1$. By the induction hypothesis, each k_1 and k_2 can be written as a product of primes, hence $k+1$ can be written as a product of primes.

(2) **The chocolate bar problem**. Prove that if you split a chocolat bar into small squares (always breaking along the lines between the squares), it always take the same number of breaks which is equal to the total number of squares minus one.

Strong induction on n, the total number of squares. The base case case $n = 1$: no breaks is needed. Suppose the assumption is true for any n, $1 \leq n \leq k$, and let $n = k+1$. After the first break we obtain two smaller bars, having k_1 and k_2 squares, respectively, where $k_1 + k_2 = k+1$. By the induction hypothesis, it takes $k_1 - 1$

breaks to split the first bar, and $k_2 - 1$ breaks to split the second bar. The total number of splits is

$$1 + (k_1 - 1) + (k_2 - 1) = k_1 - 1 + k_2 - 1 = k.$$

We have the order relation $\leq$ on $\mathbb{N}$: $-1 < 10, 3 \leq 3$.

Well-ordering principle: Any set of positive integers $\mathbb{P}$ which has at least one element has a smallest element.

True for $\mathbb{N}$ or any bounded from below non–empty subset of $\mathbb{Z}$.

THEOREM 2.1. *Well-ordering principle implies the principle of mathematical induction.*

PROOF. We prove the principle of mathematical induction assuming the WOP. So, $P(1)$ holds, and $P(k)$ true implies $P(k+1)$ true. We want to prove that $P(n)$ is true for all $n \in \mathbb{P}$. Assume that $P(n)$ does not hold for all $n \in \mathbb{P}$. Consider $S \subset \mathbb{P}$ for which $P(n)$ is false. By WOP it has the smallest element. Then $P(t-1)$ holds, and by induction step, $P(t)$, a contradiction. □

Induction, strong induction, and WOP are all equivalent.