

**AN OVERVIEW OF BASIC TOPICS IN TOPOLOGY  
AND RELATED AREAS OF GEOMETRY**

Anatole KATOK

DEPARTMENT OF MATHEMATICS THE PENNSYLVANIA STATE UNIVERSITY  
UNIVERSITY PARK, PA 16802-6401

*E-mail address:* `katok_a@math.psu.edu`



# Contents

	1
1. Basic topology	1
a. Topological spaces	1
b. Basic constructions	2
c. Continuous maps and homeomorphisms	4
d. Separation properties	5
e. Compactness	6
f. Connectedness and path connectedness	8
g. Totally disconnected spaces and Cantor sets	9
h. Topological manifolds	10
i. Orbit spaces for group actions	11
2. Metric spaces	13
a. Uniform structures	13
b. Definition and basic constructions	13
c. Maps between metric spaces	14
d. Cauchy sequences and completeness	15
e. Completion	17
f. p-adic completions of integers and rationals	18
g. Separation properties	19
h. Compact metric spaces	19
i. Spaces of continuous maps	21
3. Elementary homotopy theory	23
a. Homotopy and homotopy equivalence	23
b. Contractible spaces	23
c. Degree of a circle map	24
d. Brouwer fixed point theorem	25
e. The fundamental group	26
f. Covering spaces	27
4. Real and complex differentiable manifolds; A first introduction	29
a. Real differentiable manifolds.	29
b. Partition of unity and paracompactness	30
c. Derivatives and the tangent bundle	30
d. Differentiable maps and diffeomorphisms	31
e. Complex manifolds	31
5. Topology and geometry of surfaces	33
a. The Jordan Curve Theorem	33

b.	Euler Theorem and genus	33
c.	Gauss–Bonnet theorem	33
d.	The fundamental group	34
e.	Intersection index	34
f.	Complex structure on surfaces	34
g.	Nonorientable surfaces and surfaces with boundary	34
6.	Simplicial homology theory	36
a.	Simplicial complexes	36
7.	Differentiable manifolds	39
a.	Vector fields, flows and differential operators	39
b.	Tensor bundles	40
c.	Exterior calculus	42
d.	Transversality	43
8.	Locally compact groups and Lie groups	46

# AN OVERVIEW

## 1. Basic topology

### a. Topological spaces.

DEFINITION 1.1. A *topological space*  $(X, \mathcal{T})$  is a set  $X$  endowed with a collection  $\mathcal{T} \subset \mathcal{P}(X)$  of subsets of  $X$ , called the *topology* of  $X$ , such that

- (1)  $\emptyset, X \in \mathcal{T}$ ,
- (2) if  $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$  then  $\bigcup_{\alpha \in A} O_\alpha \in \mathcal{T}$  for any set  $A$ ,
- (3) if  $\{O_i\}_{i=1}^k \subset \mathcal{T}$  then  $\bigcap_{i=1}^k O_i \in \mathcal{T}$ ,

that is,  $\mathcal{T}$  contains  $X$  and  $\emptyset$  and is closed under union and finite intersection.

We will usually omit  $\mathcal{T}$  in the notation and will simply speak about “topological space  $X$ ” assuming the that topology has been described.

1. *Basic notions.* The sets  $O \in \mathcal{T}$  are called *open sets*, and their complements are called *closed sets*.

If  $x \in X$  then an open set containing  $x$  is called a (*open*) *neighborhood* of  $x$ .

The *closure*  $\bar{A}$  of a set  $A \subset X$  is the smallest closed set containing  $A$ , that is,  $\bar{A} := \bigcap \{C \mid A \subset C \text{ and } C \text{ closed}\}$ . A set  $A \subset X$  is called *dense* (or *everywhere dense*) if  $\bar{A} = X$ . A set  $A \subset X$  is called *nowhere dense* if  $X \setminus \bar{A}$  is everywhere dense.

A point  $x$  is said to be an *accumulation point* of  $A \subset X$  if every neighborhood of  $x$  contains infinitely many points of  $A$ .

A point  $x \in A$  is called an *interior point* of  $A$  if  $A$  contains an open neighborhood of  $x$ . The set of interior points of  $A$  is called the *interior* of  $A$  and is denoted by  $\text{Int } A$ . Thus a set is open if and only if all of its points are interior points or, equivalently  $A = \text{Int } A$ .

A point  $x$  is called a *boundary point* of  $A$  if it is neither an interior point of  $A$  nor an interior point of  $X \setminus A$ . The set of boundary points is called the *boundary* of  $A$  and is denoted by  $\partial A$ . Obviously  $\bar{A} = A \cup \partial A$ . Thus a set is closed if and only if it contains its boundary.

A sequence  $\{x_i\}_{i \in \mathbb{N}} \subset X$  is said to *converge* to  $x \in X$  if for every open set  $O$  containing  $x$  there exists  $N \in \mathbb{N}$  such that  $\{x_i\}_{i > N} \subset O$ . Any such point  $x$  is called a *limit* of the sequence.

More generally one speaks about convergence of  $\{x_i\}_{i \in \mathcal{F}} \subset X$  where  $\mathcal{F}$  is an ordered set, for example,  $\mathbb{R}$ . See Bredon, Section I.6 for a detailed discussion in great generality.

Let  $(X, \mathcal{T})$  be a topological space. A set  $D \subset X$  is said to be *dense* in  $X$  if  $\bar{D} = X$ .

The space  $X$  is said to be *separable* if it has a finite or countable dense subset.

A point  $x \in X$  is called *isolated* if the one-point set  $\{x\}$  is open.

## 2. Base of a topology.

DEFINITION 1.2. A *base* for the topology  $\mathcal{T}$  is a subcollection  $\beta \subset \mathcal{T}$  such that for every  $O \in \mathcal{T}$  and  $x \in O$  there exists  $B \in \beta$  such that  $x \in B \subset O$ .

Most topological spaces considered in analysis and geometry (but not in algebraic geometry) have *countable base*. Such topological spaces are often called *second countable*.

A *base of the neighbourhoods of a point  $x$*  is a collection  $\mathcal{B}$  of open neighbourhoods of  $x$  such that any neighbourhood of  $x$  contain an element of  $\mathcal{B}$ . If any point of a topological space has a countable base of neighborhoods the space (or the topology) is called *first countable*.

EXAMPLE 1.3. Euclidean space  $\mathbb{R}^n$  with the usual open and closed sets is a familiar example. The open balls (open balls with rational radius, open balls with rational center and radius) form a base. The latter is a countable base.

PROPOSITION 1.4. *Every topological space with a countable base is separable.*

PROOF. Pick a point in each element of a countable base. The resulting set is at most countable. It is dense since otherwise the complement to its closure would contain an element of the base.  $\square$

EXAMPLE 1.5. Consider *Zariski topology* on the real line  $\mathbb{R}$ : nonempty open sets are complements to finite sets. It is separable since it is weaker than the usual topology in  $\mathbb{R}$  (see below) but it does not have a countable base since any countable collection of open sets have nonempty intersection and thus the complement to any point in that intersection does not contain any element from this countable collection.

3. *Comparison of topologies.* A topology  $\mathcal{S}$  is said to be *finer or stronger* than  $\mathcal{T}$  if  $\mathcal{T} \subset \mathcal{S}$ , *coarser or weaker* if  $\mathcal{S} \subset \mathcal{T}$ .

There are two extreme topologies on any set: the coarsest or weakest *trivial topology* with only the whole space and the empty set being open and the strongest or finest *discrete topology* where all sets are open (and hence closed).

EXAMPLE 1.6. An uncountable set with discrete topology is an example of a first countable but not second countable topological space.

For any set  $X$  and any collection  $\mathcal{C}$  of subsets of  $X$  there exists the unique weakest topology for which all sets from  $\mathcal{C}$  are open.

Any topology weaker than a separable topology is also separable since any dense set in a stronger topology is also dense in a weaker one.

### b. Basic constructions.

1. *Induced topology.* If  $Y \subset X$  then  $Y$  can be made into a topological space in a natural way by taking the *induced topology*  $\mathcal{T}_Y := \{O \cap Y \mid O \in \mathcal{T}\}$ .

EXAMPLE 1.7. The topology induced from  $\mathbb{R}^{n+1}$  on the subset

$$\{(x_1, \dots, x_n, x_{n+1}) : \sum_{i=1}^{n+1} x_i^2 = 1\}$$

produces the (standard, or unit) *n-sphere*  $S^n$ . For  $n = 1$  it is called the (*unit*) *circle* which is sometimes also denoted by  $\mathbb{T}$ .

2. *Product topology.* If  $(X_\alpha, \mathcal{T}_\alpha)$ ,  $\alpha \in A$  are topological spaces and  $A$  is any set, then the *product topology* on  $\prod_{\alpha \in A} X_\alpha$  is the topology generated by the base  $\{\prod_{\alpha \in A} O_\alpha \mid O_\alpha \in \mathcal{T}_\alpha, O_\alpha \neq X_\alpha \text{ for only finitely many } \alpha\}$ .

EXAMPLE 1.8. The standard topology in  $\mathbb{R}^n$  coincides with the product topology on the product of  $n$  copies of the real line  $\mathbb{R}$ .

EXAMPLE 1.9. The product of  $n$  copies of the circle is called the  $n$ -torus and is usually denoted by  $\mathbb{T}^n$ . The  $n$ -torus can be naturally identified with the following subset of  $\mathbb{R}^{2n}$ :

$$\{(x_1, \dots, x_{2n}) : x_{2i-1}^2 + x_{2i}^2 = 1, i = 1, \dots, n.\}$$

with induced topology.

EXAMPLE 1.10. The product of countably many copies of the two-point space, each with discrete topology is one of the representations of the *Cantor set* (see Section 1g for a detailed discussion).

EXAMPLE 1.11. The product of countably many copies of the unit interval is called the *Hilbert cube*. It is the first interesting example of a Hausdorff space (Section 1d) “too big” to sit inside (that is, to be homeomorphic to a subset of) any Euclidean space  $\mathbb{R}^n$ .

3. *Factor topology.* Consider a topological space  $(X, \mathcal{T})$  and suppose there is an equivalence relation  $\sim$  defined on  $X$ . Then there is a natural projection  $\pi$  to the set  $\hat{X}$  of equivalence classes. The *identification space* or *factor space*  $X/\sim := (\hat{X}, \mathcal{S})$  is the topological space obtained by calling a set  $O \subset \hat{X}$  open if  $\pi^{-1}(O)$  is open, that is, taking on  $\hat{X}$  the finest topology with which  $\pi$  is continuous.

EXAMPLE 1.12. Consider the closed unit interval and equivalence relation which identifies the endpoints. Other equivalence classes are single points in the interior. The corresponding factor space is another representation of the circle.

The product of  $n$  copies of this factor space gives another representation of the  $n$ -torus.

EXERCISE 0.1. Describe the representation of the  $n$ -torus from the above example explicitly as the identification space of the unit  $n$ -cube  $I^n$ :

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1, i = 1, \dots, n\}$$

EXAMPLE 1.13. Consider the following equivalence relation in  $\mathbb{R}^{n+1} \setminus \{0\}$  :  $(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1})$  if and only if  $y_i = \lambda x_i$  for all  $i = 1, \dots, n+1$  with the same real number  $\lambda$ . The corresponding identification space is called the *real projective  $n$ -space* and is denoted by  $\mathbb{R}P(n)$ .

A similar procedure where  $\lambda$  has to be positive gives another representation of the  $n$ -sphere  $S^n$ .

EXAMPLE 1.14. Consider the following equivalence relation in  $\mathbb{C}^{n+1} \setminus \{0\}$  :  $(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1})$  if and only if  $y_i = \lambda x_i$  for all  $i = 1, \dots, n+1$  with the same complex number  $\lambda$ . The corresponding identification space is called the *complex projective  $n$ -space* and is denoted by  $\mathbb{C}P(n)$ .

A *cone* over a space  $X$  is the space obtained from identifying all points of the form  $(x, 1)$  in  $(X \times [0, 1], \text{product topology})$ .

REMARK 1.15. The notion of a “representation” of a topological space mentioned in this subsection several times has not been rigorously defined. What it means is an existence of a *homeomorphism* (Definition 1.16) between the topological spaces in question. A “natural representation” means that such a homeomorphism may be chosen in a particularly simple and natural way.

### c. Continuous maps and homeomorphisms.

DEFINITION 1.16. Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{S})$  be topological spaces.

A map  $f: X \rightarrow Y$  is said to be *continuous* if  $O \in \mathcal{S}$  implies  $f^{-1}(O) \in \mathcal{T}$ :

$f$  is an *open map* if  $O \in \mathcal{T}$  implies  $f(O) \in \mathcal{S}$ ;

$f$  is a *homeomorphism* if it is continuous and bijective with continuous inverse.

$f$  is *continuous at the point*  $x$  if for any neighborhood  $A$  of  $f(x)$  in  $Y$  the preimage  $f^{-1}(A)$  contains a neighborhood of  $x$ .

A map  $f$  from a topological space to  $\mathbb{R}$  is said to be *upper semicontinuous* if  $f^{-1}(-\infty, c) \in \mathcal{T}$  for all  $c \in \mathbb{R}$ :

*lower semicontinuous* if  $f^{-1}(c, \infty) \in \mathcal{T}$  for  $c \in \mathbb{R}$ .

EXERCISE 0.2. Proof that a map is continuous if and only if it is continuous at every point.

If there is a homeomorphism  $X \rightarrow Y$  then  $X$  and  $Y$  are said to be *homeomorphic*.

We denote by  $C^0(X, Y)$  the space of continuous maps from  $X$  to  $Y$  and write  $C^0(X)$  for  $C^0(X, \mathbb{R})$ .

A property of a topological space that is the same for any two homeomorphic spaces is said to be a *topological invariant*.

EXAMPLE 1.17. The map  $E: [0, 1] \rightarrow S^1, E(x) = \exp 2\pi ix$  establishes a homeomorphism between the interval with identified endpoints (Example 1.12) and the unit circle defined in Example 1.7.

EXERCISE 0.3. Describe a homeomorphism between the torus  $\mathbb{T}^n$  (Example 1.9) and the factor space described in Example 1.12 and the subsequent exercise.

EXERCISE 0.4. Describe a homeomorphism between the sphere  $S^n$  (Example 1.7) and the second factor space of Example 1.13.

EXERCISE 0.5. Prove that the real projective space  $\mathbb{R}P(n)$  is homeomorphic to the factor space of the sphere  $S^n$  with respect to the equivalence relation which identifies the pairs of opposite points:  $x$  and  $-x$ .

EXERCISE 0.6. Consider the equivalence relation on the closed unit ball  $\mathbb{D}^n$  in  $\mathbb{R}^n$ :

$$\{(x_1, \dots, x_n) : \sum_{i=1}^n x_i^2 \leq 1\}$$

which identifies all points of  $\partial\mathbb{D}^n = S^{n-1}$  and does nothing to interior points. Prove that the identification space is homeomorphic to  $S^n$ .

EXERCISE 0.7. Show that  $\mathbb{C}P(1)$  is homeomorphic to  $S^2$ .

Let  $Y$  be a topological space. For any collection  $\mathcal{F}$  of maps from a set  $X$  to  $Y$  there exists unique weakest topology on  $X$  which makes all maps from  $\mathcal{F}$  continuous; this is exactly the weakest topology with respect to which preimages of all open sets in  $Y$  under the maps from  $\mathcal{F}$  are open.

EXAMPLE 1.18. Zariski topology in  $\mathbb{R}^n$  or  $S^n$  is the weakest topology for which polynomials (corr. homogeneous polynomials) are continuous functions.

The notion of a topological space is very general. It is very useful as an “umbrella” concept which allows to use certain elements of geometric language and the way of thinking in the broad variety of vastly different situations. However because of such generality and elasticity of the concept very little can be said about topological spaces in full generality. Now we proceed to the discussion of various properties (and corresponding special classes) of topological spaces which reflect different aspects of the familiar examples. All properties introduced in the rest of this section are topological invariants.

In discussing these properties among other things we will look how they behave under continuous maps and under the three basic constructions: inducing, products and factors.

**d. Separation properties.** These properties give one of several natural ways of measuring measure how fine is a given topology.

DEFINITION 1.19. Let  $(X, \mathcal{T})$  be a topological space.

- (1)  $(X, \mathcal{T})$  is called a *(T1) space* if any point is a closed set. Equivalently for any pair of points  $x_1, x_2 \in X$  there exists a neighborhood of  $x_1$  which does not contain  $x_2$ .
- (2)  $(X, \mathcal{T})$  is called a *(T2) or Hausdorff space* if for any two  $x_1, x_2 \in X$  there exist  $O_1, O_2 \in \mathcal{T}$  such that  $x_i \in O_i$  and  $O_1 \cap O_2 = \emptyset$ .
- (3)  $(X, \mathcal{T})$  is called a *(T4) or normal space* if for any two closed  $X_1, X_2 \subset X$  there exist  $O_1, O_2 \in \mathcal{T}$  such that  $X_i \subset O_i$  and  $O_1 \cap O_2 = \emptyset$ .

It follows immediately from the definition of the induced topology that any of the above separation properties is inherited by the induced topology on any subset.

EXERCISE 0.8. Prove that in a (T1) space any sequence has no more than one limit.

EXERCISE 0.9. Prove that the product of two (T1) (corr. Hausdorff) spaces is a (T1) (corr. Hausdorff) space.

REMARK 1.20. We will see later (Section 1i) that even very naturally defined equivalence relations in nice spaces may produce factor spaces with widely varying separation properties.

A useful consequence of normality is the following extension result:

THEOREM 1.21. (*Bredon, Theorem 10.4*) *If  $X$  is a normal topological space,  $Y \subset X$  closed, and  $f: Y \rightarrow \mathbb{R}$  continuous, then there is a continuous extension of  $f$  to  $X$ .*

Most natural topological spaces which appear in analysis and geometry (but not in various branches of algebra) are normal.

EXAMPLE 1.22. Zariski topology on the real line (Example 1.5) is obviously (T1) (the complement to any point is open) but not Hausdorff (any two nonempty open sets have nonempty intersection).

An equivalent way to define this topology is to say that closed sets are sets of zeroes of polynomials. This is the simplest example of a *Zariski topology*. Other simple examples are defined in Euclidean space  $\mathbb{R}^n$  where similarly closed sets are sets of zeroes of polynomials or on the unit sphere  $S^n \subset \mathbb{R}^{n+1}$  where closed sets are defined as sets of zeroes of homogeneous polynomials in  $n + 1$  variables. One can also define Zariski topologies on the real and complex projective spaces  $\mathbb{R}P(n)$  and  $\mathbb{C}P(n)$  (Example 1.13, Example 1.14) via zero sets of homogeneous polynomials in  $n + 1$  real and complex variables correspondingly.

Zariski topologies play an important role in algebraic geometry and the theory of algebraic groups.

**e. Compactness.**  $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$  is called an *open cover* of  $X$  if  $X = \bigcup_{\alpha \in A} O_\alpha$ , and a *finite open cover* if  $A$  is finite.

DEFINITION 1.23. The space  $(X, \mathcal{T})$  is called *compact* if every open cover has a finite subcover;

$(X, \mathcal{T})$  *locally compact* if every point has a neighborhood with compact closure;

$(X, \mathcal{T})$  *sequentially compact* if every sequence has a convergent subsequence.

$X$  is called  $\sigma$ -compact if it is a countable union of compact sets.

EXERCISE 0.10. Prove that Zariski topology on  $\mathbb{R}$  is compact but not sequentially compact.

PROPOSITION 1.24. *A closed subset of a compact set is compact.*

PROOF. If  $K$  is compact,  $C \subset K$  is closed, and  $\Gamma$  is an open cover for  $C$  then  $\Gamma \cup \{K \setminus C\}$  is an open cover for  $K$ , hence has a finite subcover  $\Gamma' \cup \{K \setminus C\}$ , so  $\Gamma'$  is a finite subcover (of  $\Gamma$ ) for  $C$ .  $\square$

PROPOSITION 1.25. *A compact subset of a Hausdorff space is closed.*

PROOF. If  $X$  is Hausdorff and  $C \subset X$  compact fix  $x \in X \setminus C$  and for each  $y \in C$  take neighborhoods  $U_y$  of  $y$  and  $V_y$  of  $x$  such that  $U_y \cap V_y = \emptyset$ . The cover  $\bigcup_{y \in C} U_y \supset C$  has a finite subcover  $\{U_{x_i} \mid 0 \leq i \leq n\}$  and hence  $N_x := \bigcap_{i=0}^n V_{y_i}$  is a neighborhood of  $x$  disjoint from  $C$ . Thus  $X \setminus C = \bigcup_{x \in X \setminus C} N_x$  is open and  $C$  is closed.  $\square$

PROPOSITION 1.26. *A compact Hausdorff space is normal.*

PROOF. First we show that a closed set  $K$  and a point  $p \notin K$  can be separated by open sets. For  $x \in K$  there are open sets  $O_x, U_x$  such that  $x \in O_x, p \in U_x$  and  $O_x \cap U_x = \emptyset$ . Since  $K$  is compact there is a finite subcover  $O := \bigcup_{i=1}^n O_{x_i} \supset K$ , and  $U := \bigcap_{i=1}^n U_{x_i}$  is an open set containing  $p$  disjoint from  $O$ . Now suppose  $K, L$  are closed sets and for  $p \in L$  consider open disjoint sets  $O_p \supset K, U_p \ni p$ . By compactness of  $L$  there is a finite subcover  $U := \bigcup_{j=1}^m U_{p_j} \supset L$  and  $O := \bigcap_{j=1}^m O_{p_j} \supset K$  is an open set disjoint from  $U$ .  $\square$

A collection of sets is said to have the *finite intersection property* if every finite subcollection has nonempty intersection.

PROPOSITION 1.27. *A collection of compact sets with the finite intersection property has nonempty intersection.*

PROOF. It suffices to show that in a compact space every collection of closed sets with the finite intersection property has nonempty intersection. To that end consider a collection of closed sets with empty intersection. Their complements form an open cover. Since it has a finite subcover the finite intersection property does not hold.  $\square$

DEFINITION 1.28. The *one-point compactification* of a noncompact Hausdorff space  $(X, \mathcal{T})$  is  $\hat{X} := (X \cup \{\infty\}, \mathcal{S})$ , where  $\mathcal{S} := \mathcal{T} \cup \{(X \cup \{\infty\}) \setminus K \mid K \subset X \text{ compact}\}$ .

EXERCISE 0.11. Show that the one-point compactification of a Hausdorff space  $X$  is a compact Hausdorff space with  $X$  as a dense subset.

THEOREM 1.29 (Tychonoff Theorem). (*Bredon, Theorem 8.9*) *The product of compact spaces is compact.*

This result is useful in many situations because often a useful topology can be viewed as a product topology or is induced by a product topology. An obvious example is the topology of pointwise convergence.

We will prove a particular case of Tychonoff Theorem for the product of two (and hence finitely many) spaces.

PROOF. Consider an open cover  $\mathcal{C}$  of the product of two compact topological spaces  $X$  and  $Y$ . Without loss of generality we can assume that every element of  $\mathcal{C}$  is the product of open subsets in  $X$  and  $Y$ . Since for each  $x \in X$  the subset  $\{x\} \times Y$  with induced topology is homeomorphic to  $Y$  and hence compact, one can find a finite subcollection  $\mathcal{C}_x \subset \mathcal{C}$  which covers  $\{x\} \times Y$ .

Let for  $(x, y) \in X \times Y$ ,  $\pi_1(x, y) = x$ . Let  $U_x = \bigcap_{C \in \mathcal{C}_x} \pi_1(C)$ ; this is an open neighborhood of  $x$  and since elements of  $\mathcal{C}_x$  are products,  $\mathcal{C}_x$  covers  $U_x \times Y$ . The sets  $U_x$ ,  $x \in X$  form an open cover of  $X$ . By compactness of  $X$  there is a finite subcover, say  $\{U_{x_1}, \dots, U_{x_k}\}$ . Then the union of collections  $\mathcal{C}_{x_1}, \dots, \mathcal{C}_{x_k}$  form an open cover of  $X \times Y$ .  $\square$

PROPOSITION 1.30. *The image of a compact set under a continuous map is compact.*

PROOF. If  $C$  is compact and  $f: C \rightarrow Y$  continuous and surjective then any open cover  $\Gamma$  of  $Y$  induces an open cover  $f_*\Gamma := \{f^{-1}(O) \mid O \in \Gamma\}$  of  $C$  which by compactness has a finite subcover  $\{f^{-1}(O_i) \mid i = 1, \dots, n\}$ . By surjectivity  $\{O_i\}_{i=1}^n$  is a cover for  $Y$ .  $\square$

A useful application of the notions of continuity, compactness, and separation is the following result, sometimes referred to as *invariance of domain*:

PROPOSITION 1.31. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

PROOF. Suppose  $X$  is compact,  $Y$  Hausdorff,  $f: X \rightarrow Y$  bijective and continuous, and  $O \subset X$  open. Then  $C := X \setminus O$  is closed, hence compact, and  $f(C)$  is compact, hence closed, so  $f(O) = Y \setminus f(C)$  (by bijectivity) is open.  $\square$

### f. Connectedness and path connectedness.

DEFINITION 1.32. A topological space  $(X, \mathcal{T})$  is said to be *connected* if no two disjoint open sets cover  $X$ ;

$(X, \mathcal{T})$  is said to be *path connected* if for any two points  $x_0, x_1 \in X$  there exists a continuous curve  $c: [0, 1] \rightarrow X$  with  $c(i) = x_i$ ,  $i = \{0, 1\}$ .

#### 1. Products and factors.

THEOREM 1.33. *The product of two connected topological spaces is connected.*

PROOF. Suppose  $X, Y$  are connected and assume that  $X \times Y = A \cup B$  where  $A$  and  $B$  are open and  $A \cap B = \emptyset$ . Then either  $A = X_1 \times Y$  for some open  $X_1 \subset X$  or there exists  $x \in X$  such that  $\{x\} \times Y \cap A \neq \emptyset$  and  $\{x\} \times Y \cap B \neq \emptyset$ .

The former case is impossible since then  $B = (X \setminus X_1) \times Y$  and hence  $X = X_1 \cup (X \setminus X_1)$  is not connected.

In the latter case  $Y = \pi_2(\{x\} \times Y \cap A) \cup \pi_2(\{x\} \times Y \cap B)$ , that is,  $Y$  is the union of two disjoint open sets, hence not connected. Here for  $(x, y) \in X \times Y$ ,  $\pi_2(x, y) = y$ . Obviously  $\pi_2$  restricted to  $\{x\} \times Y$  is a homeomorphism.  $\square$

EXERCISE 0.12. Prove that the product of two path connected spaces is path connected.

The following property follows immediately from the definition of the factor topology

PROPOSITION 1.34. *Any factor space of a connected topological space is connected.*

#### 2. Invariance under continuous maps.

THEOREM 1.35. *A continuous image of a connected space  $X$  is connected.*

PROOF. If the image is decomposed into the union of two disjoint open sets, the preimages of these sets which are open by continuity would give a similar decomposition for  $X$ .  $\square$

EXERCISE 0.13. Prove that a path-connected space is connected.

REMARK 1.36. The converse is false as is shown by the union of the graph of  $\sin 1/x$  and  $\{0\} \times [-1, 1]$  in  $\mathbb{R}^2$ .

3. *Connected subsets and connected components.* A subset of a topological space is *connected (path connected)* if it is a connected (path connected) space in induced topology.

A *connected component* of a topological space  $X$  is a maximal connected subset of  $X$ ;

A *path connected component* of  $X$  is a maximal path connected subset of  $X$ ;

PROPOSITION 1.37. *Closure of a connected subset  $Y \subset X$  is connected.*

PROOF. If  $\bar{Y} = Y_1 \cup Y_2$  where  $Y_1, Y_2$  are open and  $Y_1 \cap Y_2 = \emptyset$  then since the set  $Y$  is dense in its closure  $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$  with both  $Y \cap Y_1$  and  $Y \cap Y_2$  open in induced topology and nonempty.  $\square$

COROLLARY 1.38. *Connected components are closed.*

**PROPOSITION 1.39.** *Union of two connected subsets  $Y_1, Y_2 \subset X$  such that  $Y_1 \cap Y_2 \neq \emptyset$ , is connected.*

**PROOF.** We will argue by contradiction. Assume that  $Y_1 \cap Y_2$  is the disjoint union of open sets  $Z_1$  and  $Z_2$ . If  $Z_1 \supset Y_1$  then  $Y_2 = Z_2 \cup (Z_1 \cap Y_2)$  and hence not connected. Similarly it is impossible that  $Z_2 \supset Y_1$ . Thus  $Y_1 \cap Z_i \neq \emptyset$ ,  $i = 1, 2$  and hence  $Y_1 = (Y_1 \cap Z_1) \cup (Y_1 \cap Z_2)$  and hence not connected.  $\square$

4. *Decomposition into connected and path connected components.* For any topological space there is a unique *decomposition into connected components* and a unique *decomposition into path connected components* elements of these decompositions are equivalence classes of the following two equivalence relations correspondingly:

(i)  $x$  is equivalent to  $y$  if there exists a connected subset  $Y \subset X$  which contains  $x$  and  $y$ . In order to show that the equivalence classes are indeed connected components one needs to prove that they are connected. For, if  $A$  is an equivalence class, assume that  $A = A_1 \cup A_2$  where  $A_1$  and  $A_2$  are disjoint and open. Pick  $x_1 \in A_1$  and  $x_2 \in A_2$  and find a closed connected set  $A_3$  which contains both points. But then  $A \subset (A_1 \cup A_3) \cup A_2$  which is connected by Proposition 1.39. Hence  $A = (A_1 \cup A_3) \cup A_2$  and  $A$  is connected.

(ii)  $x$  is equivalent to  $y$  if there exists a continuous curve  $c: [0, 1] \rightarrow X$  with  $c(0) = x$ ,  $c(1) = y$

Notice that the closure of a path connected subset may not be path connected.

**g. Totally disconnected spaces and Cantor sets.** On the opposite end from connected spaces lie those spaces which do not have connected nontrivial connected subsets at all. A topological space  $(X, \mathcal{T})$  is said to be *totally disconnected* if every point is a connected component. In other words, only connected subsets of  $X$  are single points.

Discrete topologies (all points are open) give trivial examples of totally disconnected topological spaces. Another example is the set  $\{0, 1, 1/2, 1/3, \dots\}$  with topology induced from the real line. More complicated examples of compact totally disconnected space where isolated points are dense can be easily constructed.

Two very different examples are the set of rational numbers  $\mathbb{Q} \subset \mathbb{R}$  with induced topology which is not locally compact, and the (standard, middle-third) *Cantor set*  $C$ :

$$\{x \in \mathbb{R} : x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}, x_i \in \{0, 2\}, i = 1, 2, \dots\}.$$

The Cantor set is compact Hausdorff with countable base (it is a closed subset of  $[0, 1]$ ), and perfect (no isolated points). As it turns out it is a universal model for totally disconnected spaces with such properties.

**THEOREM 1.40.** *Any compact Hausdorff perfect topological space with countable base which is totally disconnected is homeomorphic to the Cantor set  $C$ .*

This theorem will be proven later using machinery of metric spaces (See Theorem 2.24). For now we restrict ourselves to certain examples and particular cases which will be used in the general proof.

**EXAMPLE 1.41.** The countable product of two point spaces with discrete topology is homeomorphic to the Cantor set. To see that identify each factor in the product with  $\{0, 2\}$  and consider the map  $(x_1, x_2, \dots) \rightarrow \sum_{i=1}^{\infty} \frac{x_i}{3^i}$  where  $x_i \in \{0, 2\}$ ,  $i = 1, 2, \dots$ . This map is a homeomorphism between the product and the Cantor set.

EXAMPLE 1.42. The product of two (and hence of any finite number) of Cantor sets is homeomorphic to the Cantor set. This follows immediately since the product of two countable products of two point spaces can be presented as such a product by mixing coordinates.

EXERCISE 0.14. Show that the product of countably many copies of the Cantor set is homeomorphic to the Cantor set.

PROPOSITION 1.43. *Any compact perfect totally disconnected subset  $A$  of the real line is homeomorphic to the Cantor set.*

PROOF. The set  $A$  is bounded by compactness and nowhere dense (does not contain any interval) since it is totally disconnected. Let  $m = \inf A$  and  $M = \sup A$ . We will outline a construction of a strictly monotone function  $F : [0, 1] \rightarrow [m, M]$  such that  $F(C) = A$ . The set  $[m, M] \setminus A$  is the union of countably many disjoint intervals without common ends (since  $A$  is perfect). Take of the intervals whose length is maximal (there are finitely many of them); denote it by  $I$ . Define  $F$  on the interval  $I$  as the increasing linear map whose image is the interval  $[1/3, 2/3]$ . Consider longest intervals  $I_1$  and  $I_2$  to the right and to the left to  $I$ . Map them linearly onto  $[1/9, 2/9]$  and  $[7/9, 8/9]$  correspondingly. The complement  $[m, M] \setminus (I_1 \cup I \cup I_2)$  consists of four intervals which are mapped linearly onto the middle third intervals of  $[0, 1] \setminus ([1/9, 2/9] \cup [1/3, 2/3] \cup [7/9, 8/9])$  and so on by induction. Eventually one obtains a strictly monotone bijective map  $[m, M] \setminus A \rightarrow [0, 1] \setminus C$  which by continuity is extended to the desired homeomorphism.  $\square$

EXERCISE 0.15. Prove that the product of countably many finite sets with discrete topology is homeomorphic to the Cantor set.

**h. Topological manifolds.** At the other end of the scale from totally disconnected spaces are the most important objects of algebraic and differential topology: the spaces which locally look as a Euclidean space.

DEFINITION 1.44. A *topological manifold* is a Hausdorff space  $X$  with a countable base for the topology such that every point is contained in an open set homeomorphic to a ball in  $\mathbb{R}^n$ . A pair  $(U, h)$  of such a neighborhood and a homeomorphism  $h : U \rightarrow B \subset \mathbb{R}^n$  is called a *chart* or a system of *local coordinates*.

A *topological manifold with boundary* is a Hausdorff space  $X$  with a countable base for the topology such that every point is contained in an open set homeomorphic to an open set in  $\mathbb{R}^{n-1} \times [0, \infty)$ .

An open subset of a topological manifold is a topological manifold.

If  $X$  is connected then  $n$  is constant. In this case it is called the *dimension* of the topological manifold. Invariance of the dimension (in other words, the fact that  $\mathbb{R}^n$  or open sets in those for different  $n$  are not homeomorphic) is one of the basic and not trivial facts of topology.

Path connectedness and connectedness are equivalent for topological manifolds.

EXAMPLE 1.45. The  $n$ -sphere  $S^n$ , the  $n$ -torus  $\mathbb{T}^n$  and the real projective  $n$ -space  $\mathbb{R}P(n)$  are examples of  $n$  dimensional connected topological manifolds; the complex projective  $n$ -space  $\mathbb{C}P(n)$  is a topological manifold of dimension  $2n$ .

EXAMPLE 1.46. Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuously differentiable function and let  $c$  be a noncritical value of  $F$ , that is, there are no critical points at which the value of  $F$  is equal to  $c$ . Then  $F^{-1}(c)$  (if nonempty) is a topological manifold of dimension  $n - 1$ . Similarly  $F^{-1}((-\infty, c])$  is an  $n$ -dimensional topological manifold with boundary.

**i. Orbit spaces for group actions.** An important class of factor spaces appears when the equivalence relation is given by an action of a group  $X$  by homeomorphisms of a topological space  $X$ . Equivalence classes are *orbits* of such a group.

The identification space in this case is denoted by  $X/G$  and called the *quotient* of  $X$  by  $G$ .

The main issue here is that in general the identification space even for a nice looking group of homeomorphisms of a good (for example, locally compact normal with countable base) topological space may not have good separation properties. (T1) property for the identification space is easy to decide: every orbit of the action must be closed. On the other hand, there does not seem to be a natural necessary and sufficient condition for the factor space to be Hausdorff. Some useful sufficient conditions will appear in the context of metric spaces.

Still, lots of important spaces appear naturally as such identification spaces.

EXAMPLE 1.47. Consider the natural action of the integer lattice  $\mathbb{Z}^n$  by translations in  $\mathbb{R}^n$ . The orbits are translates of the integer lattice  $\mathbb{Z}^n$ . The factor space is homeomorphic to the torus  $\mathbb{T}^n$ .

An even simpler situation produces a very interesting example.

EXAMPLE 1.48. Consider the action of the cyclic group of two elements on the sphere  $S^n$  generated by the central symmetry:  $Ix = -x$ . The factor space is naturally identified with the real projective space  $\mathbb{R}P(n)$ .

EXERCISE 0.16. Consider the cyclic group of order  $q$  generated by the rotation of the circle by the angle  $2\pi/q$ . Prove that the identification space is homeomorphic to the circle.

EXERCISE 0.17. Consider the cyclic group of order  $q$  generated by the rotation of the plane  $\mathbb{R}^2$  around the origin by the angle  $2\pi/q$ . Prove that the identification space is homeomorphic to  $\mathbb{R}^2$ .

EXERCISE 0.18. [Hopf fibration] Consider the unit sphere in  $\mathbb{C}^2$ :

$$\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$$

and the action  $H$  of the circle on it by the scalar multiplication: For  $\lambda \in S^1$  let  $H_\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2)$ .

Prove that the identification space is homeomorphic to  $S^2$ .

Now we proceed to show some examples where the identification spaces are not so nice.

EXAMPLE 1.49. Consider the following action  $A$  of  $\mathbb{R}$  on  $\mathbb{R}^2$ : for  $t \in \mathbb{R}$  let  $A_t(x, y) = (x + y, y)$ . The orbit space can be identified with the union of two coordinate axes: every point on the  $x$ -axis is fixed and every orbit away from it intersects the  $y$ -axis at a single point. However the factor topology is weaker than the topology induced from  $\mathbb{R}^2$  would be. Neighborhoods of the points on the  $y$ -axis are ordinary but any neighborhood of a point

on the  $x$ -axis includes a small open interval of the  $y$ -axis around the origin. Thus points on the  $x$ -axis cannot be separated by open neighborhoods and the space is (T1) (since orbits are closed) but not Hausdorff.

An even weaker but still nontrivial separation property appears in the following example.

EXAMPLE 1.50. Consider the action of  $\mathbb{Z}$  on  $\mathbb{R}$  generated by the map  $x \rightarrow 2x$ . The factor space can be identified with the union of the circle and an extra point  $p$ . Induced topology on the circle is standard. However, the only open set which contains  $p$  is the whole space!

Finally let us point out that if all orbits of an action are dense then the factor topology is obviously trivial: no invariant open sets other than  $\emptyset$  and the whole space. Here is a concrete example.

EXAMPLE 1.51. Consider the action  $T$  of  $\mathbb{Q}$ , the additive group of rational number on  $\mathbb{R}$  by translations: for  $r \in \mathbb{Q}$  and  $x \in \mathbb{R}$  let  $T_r(x) = x + r$ . The orbits are translations of  $\mathbb{Q}$ , hence dense. Thus the factor topology is trivial.

## 2. Metric spaces

**a. Uniform structures.** The general notion of topology does not allow to compare neighborhoods of different points. Such a comparison is quite natural in various geometric contexts. The most general setting for such a comparison is that of a *uniform structure*. The most common and natural way for a uniform structure to appear is via a metric, although there are other important situations such as topological groups. Also as it turns out a Hausdorff compact space carries a natural uniform structure which in the separable case can be recovered from any metric generating the topology.

### b. Definition and basic constructions.

#### 1. Definition of a metric space.

DEFINITION 2.1. If  $X$  is a set then  $d: X \times X \rightarrow \mathbb{R}$  is called a *metric* if

- (1)  $d(x, y) = d(y, x)$  (*symmetry*),
- (2)  $d(x, y) = 0 \Leftrightarrow x = y$  (*positivity*),
- (3)  $d(x, y) + d(y, z) \geq d(x, z)$  (*the triangle inequality*).

If  $d$  is a metric then  $(X, d)$  is called a *metric space*. The set  $B(x, r) := \{y \in X \mid d(x, y) < r\}$  is called the (*open*)  $r$ -ball around  $x$ . The set  $B_c(x, r) = \{y \in X \mid d(x, y) \leq r\}$  is called the *closed*  $r$ -ball around  $x$ .

2. *Metric topology.*  $O \subset X$  is called *open* if for every  $x \in O$  there exists  $r > 0$  such that  $B(x, r) \subset O$ . It follows immediately from the definition that open sets satisfy Definition 1.1. Topology thus defined is sometimes called the *metric topology* or *topology generated by the metric*  $d$ . Naturally different metrics may define the same topology.

Notice that the closed ball  $B_c(x, r)$  contains the closure of the open ball  $B(x, r)$  but may not coincide with it (Just consider the integers with the the standard metric:  $d(m, n) = |m - n|$ .)

Open balls as well as balls of rational radius or balls of radius  $r_n$ ,  $n = 1, 2, \dots$  where  $r_n$  converges to zero form a base of the metric topology. Thus metric topology is always first countable. It is second countable if the space is separable since balls of the any of the above form centered at the points of a dense set also form a basis of the topology.

Thus the closure of  $A \subset X$  has the form  $\bar{A} = \{x \in X \mid \forall r > 0 \quad B(x, r) \cap A \neq \emptyset\}$ . For any closed set  $A$  and any point  $x \in X$  the *distance from  $x$  to  $A$* ,

$$d(x, A) := \inf_{y \in A} d(x, y)$$

is defined. It is positive if and only if  $x \in X \setminus A$ .

Let  $\varphi: [0, \infty] \rightarrow \mathbb{R}$  be a nondecreasing, continuous, concave function such that  $\varphi^{-1}(\{0\}) = \{0\}$ . If  $(X, d)$  is a metric space than  $\varphi \circ d$  is another metric on  $d$  which generates the same topology.

It is interesting to notice what happens if a function  $d$  as in Definition 2.1 does not satisfy symmetry or positivity. In the former case it can be symmetrized producing a metric  $d_S(x, y) := \max(d(x, y), d(y, x))$ . In the latter by the symmetry and triangle inequality the condition  $d(x, y) = 0$  defines an equivalence relation and a genuine metric is defined in the space of equivalence classes. Note that some of the most important notions in analysis such as spaces  $L^p$  of functions on a measure space are actually such factor spaces.

For metric spaces the converse to Proposition 1.4 is also true.

PROPOSITION 2.2. *Every separable metric space has countable base.*

PROOF. By the triangle inequality every open ball contains an open ball around a point of a dense set. Thus for a separable spaces balls of rational radius around points of a countable dense set form a base of the metric topology.  $\square$

3. *Induced metric.* Any subset  $A$  of a metric space  $X$  is a metric space with an *induced metric*  $d_A$ , the restriction of  $d$  to  $A \times A$ . *Diameter* of a set in a metric space is the supremum of distances between its points; it is often denoted by  $\text{diam } A$ . The set  $A$  is called *bounded* if it has finite diameter.

4. *Metrics in product spaces.* For the product of finitely many metric spaces there are various natural ways to introduce a metric. Let  $\varphi : ([0, \infty])^n \rightarrow \mathbb{R}$  be a continuous concave function nondecreasing in each variable and such that  $\varphi^{-1}(\{0\}) = \{(0, \dots, 0)\}$

Given metric spaces  $(X_i, d_i)$ ,  $i = 1, \dots, n$  let

$$d^\varphi := \varphi(d_1, \dots, d_n) : (X_1 \times \dots \times X_n) \times (X_1 \times \dots \times X_n) \rightarrow \mathbb{R}.$$

EXERCISE 0.1. Prove that  $d^\varphi$  defines a metric on  $X_1 \times \dots \times X_n$  which generates the product topology.

Here are examples which appear most often:

The *maximum metric* corresponds to

$$\varphi(t_1, \dots, t_n) = \max(t_1, \dots, t_n).$$

The  $l^p$  metric for  $1 \leq p < \infty$  corresponds to

$$\varphi(t_1, \dots, t_n) = (t_1^p + \dots + t_n^p)^{1/p}.$$

Two particularly important cases of the latter are  $t = 1$  and  $t = 2$ ; the latter produces Euclidean metric in  $\mathbb{R}^n$  from the standard (absolute value) metrics on  $n$  copies of  $\mathbb{R}$ .

For the countable product of metric spaces various metrics generating the product topology can also be introduced. One class of such metrics can be produced as follows. Let  $\varphi : [0, \infty] \rightarrow \mathbb{R}$  be as above and let  $a_1, a_2, \dots$  be a sequence of positive numbers such that the series  $\sum_{n=1}^{\infty} a_n$  converges. Given metric spaces  $(X_1, d_1), (X_2, d_2) \dots$  consider the metric  $d$  on  $\prod_{n=1}^{\infty} X_n$  defined as

$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sum_{n=1}^{\infty} a_n \varphi(d_n(x_n, y_n)).$$

EXERCISE 0.2. Prove that  $d$  is really a metric and that the corresponding metric topology coincides with the product topology.

**c. Maps between metric spaces.** Let  $(X, d), (Y, \text{dist})$  be metric spaces. A map  $f: X \rightarrow Y$  is said to be *uniformly continuous* if for all  $\epsilon > 0$  there is a  $\delta > 0$  such that for all  $x, y \in X$  with  $d(x, y) < \delta$  we have  $\text{dist}(f(x), f(y)) < \epsilon$ . A uniformly continuous bijection with uniformly continuous inverse is called a *uniform homeomorphism*. It is not hard to show that a uniformly continuous map from a subset of a metric space uniquely extends to the closure.

A family  $\mathcal{F}$  of maps  $X \rightarrow Y$  is said to be *equicontinuous* if for every  $x \in X$  and  $\epsilon > 0$  there is a  $\delta > 0$  such that  $d(x, y) < \delta$  implies  $\text{dist}(f(x), f(y)) < \epsilon$  for all  $y \in X$  and  $f \in \mathcal{F}$ .

A map  $f: X \rightarrow Y$  is said to be *Hölder continuous* with exponent  $\alpha$ , or  $\alpha$ -*Hölder*, if there exist  $C, \epsilon > 0$  such that  $d(x, y) < \epsilon$  implies  $d(f(x), f(y)) \leq C(d(x, y))^\alpha$ , *Lipschitz continuous* if it is 1-Hölder, and *biLipschitz* if it is Lipschitz and has a Lipschitz inverse.

It is called an *isometry* if  $d(f(x), f(y)) = d(x, y)$  for all  $x, y \in X$ .

Two metric spaces are *uniformly equivalent* if there exists a homeomorphism between the spaces which is uniformly continuous together with its inverse.

EXAMPLE 2.3. The open interval and the real line are homeomorphic but not uniformly equivalent.

Metric spaces are *Hölder equivalent* if there there exists a homeomorphism between the spaces which is Hölder together with its inverse.

Metric spaces are *Lipschitz equivalent* if there exists a biLipschitz homeomorphism between the spaces.

EXAMPLE 2.4. Consider the standard middle-third Cantor set  $C$  and the subset  $C_1$  of  $[0, 1]$  obtained by a similar procedure but with taking away at every step the open interval in the middle of one half of the length. These two sets are Hölder equivalent but not Lipschitz equivalent.

EXERCISE 0.3. Find a Hölder homeomorphism with Hölder inverse in the previous example.

EXERCISE 0.4. Prove that the identity map of the product space is biLipschitz homeomorphism between the space provided with the maximal metric and with any  $l^p$  metric.

Metric spaces are *isometric* if there exists a homeomorphism between the spaces which is an isometry (in this case the inverse is obviously an isometry too.)

EXAMPLE 2.5. The unit square (open or closed) is Lipschitz equivalent to the unit disc (corr. open or closed) but not isometric to it.

EXERCISE 0.5. Consider the unit circle with the metric induced from the  $\mathbb{R}^2$  and the unit circle with the angular metric. Prove that these two metric spaces are Lipschitz equivalent but not isometric.

#### d. Cauchy sequences and completeness.

##### 1. Definition and basic properties.

DEFINITION 2.6. A sequence  $\{x_i\}_{i \in \mathbb{N}}$  is called a *Cauchy sequence* if for all  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  such that  $d(x_i, x_j) < \epsilon$  whenever  $i, j \geq N$ ;  $X$  is said to be *complete* if every Cauchy sequence converges.

PROPOSITION 2.7. *Any metric space uniformly equivalent to a complete space is complete.*

PROOF. A uniformly continuous map obviously takes Cauchy sequence into Cauchy sequences.  $\square$

EXERCISE 0.6. Prove that an open interval, an open half-line and the whole line are mutually not uniformly equivalent.

PROPOSITION 2.8. *A subset  $A$  of a complete metric space  $X$  is a complete metric space with respect to induced metric if and only if it is closed.*

PROOF. For a closed  $A \in X$  the limit of any Cauchy sequence in  $A$  belong to  $A$ . If  $A$  is not closed take a sequence in  $A$  converging to a point in  $\bar{A} \setminus A$ . It is Cauchy but does not converge in  $A$ .  $\square$

The following basic property of complete spaces is used in the next two theorems.

PROPOSITION 2.9. *Let  $A_1 \supset A_2 \supset \dots$  be a nested sequence of closed sets in a complete metric space, such that  $\text{diam } A_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\bigcap_{n=1}^{\infty} A_n$  is a single point.*

PROOF. Since  $\text{diam } A_n \rightarrow 0$  the intersection cannot contain more than one point. Take a sequence  $x_n \in A_n$ . It is Cauchy since  $\text{diam } A_n \rightarrow 0$ . Its limit  $x$  belongs to  $\bar{A}_n$  for any  $n$ . By closeness of the sets  $x \in A_n$  for any  $n$ .  $\square$

### 2. The Baire category theorem.

THEOREM 2.10 (Baire Category Theorem). *In a complete metric space a countable intersection of open dense sets is dense. The same holds for a locally compact Hausdorff space.*

PROOF. If  $\{O_i\}_{i \in \mathbb{N}}$  are open and dense in  $X$  and  $\emptyset \neq B_0 \subset X$  is open then inductively choose a ball  $B_{i+1}$  of radius at most  $\epsilon/i$  such that  $\bar{B}_{i+1} \subset O_{i+1} \cap B_i$ . The centers converge by completeness, so  $\emptyset \neq \bigcap_i \bar{B}_i \subset B_0 \cap \bigcap_i O_i$ . For locally compact Hausdorff spaces take  $B_i$  open with compact closure and use the finite intersection property.  $\square$

Baire Theorem motivates the following definition. If we want to measure massiveness of sets in a topological or in particular metric space we may assume that nowhere dense are small and their complements are massive. the next step is to consider countable unions of nowhere dense sets; such sets are called *sets of first (Baire) category*. The Baire category theorem asserts that at least for complete metric spaces such sets still can be viewed as small since they cannot fill any open set.

### 3. Minimality of the Cantor set.

THEOREM 2.11. *Any uncountable separable complete metric space  $X$  contains a closed subset homeomorphic to the Cantor set.*

PROOF. First consider the following subset

$$X_0 := \{x \in X \mid \text{any neighbourhood of } x \text{ contains uncountably many points}\}$$

Notice that the set  $X_0$  is perfect, that is, it is closed and contains no isolated points.

LEMMA 2.12. *The set  $X \setminus X_0$  is countable.*

PROOF. (of the lemma) For each point  $x \in X \setminus X_0$  find a neighborhood from a countable base which contains at most countably many points (Proposition 2.2). Thus  $X \setminus X_0$  is covered by at most countably many sets each containing at most countably many points.  $\square$

Thus the theorem would follow from the following fact.

**PROPOSITION 2.13.** *Any perfect complete metric space  $X$  contains a closed subset homeomorphic to the Cantor set.*

**PROOF.** (of the proposition) Pick two points  $x_0 \neq x_1$  in  $X$  and let  $d_0 := d(x_0, x_1)$ . Let  $X_i := \overline{B(x_i, (1/4)d_0)}$ ,  $i = 0, 1$  and  $C_1 := X_0 \cup X_1$ .

Then pick two different points  $x_{i,0}, x_{i,1} \in \text{Int } X_i$ ,  $i = 0, 1$ . Such choices are possible because any open set in  $X$  contains infinitely many points. Notice that  $d(x_{i,0}, x_{i,1}) \leq (1/2)d_0$ . Let  $Y_{i_1, i_2} := \overline{B(x_{i_1, i_2}, (1/4)d(x_{i_1, 0}, x_{i_1, 1}))}$ ,  $i_1, i_2 = 0, 1$ ,  $X_{i_1, i_2} := Y_{i_1, i_2} \cap C_1$  and  $C_2 = X_{0,0} \cup X_{0,1} \cup X_{1,0} \cup X_{1,1}$ . Notice that  $\text{diam}(X_{i_1, i_2}) \leq d_0/2$ .

Proceed by induction. Having constructed

$$C_n = \bigcup_{i_1, \dots, i_n \in \{0,1\}} X_{i_1, \dots, i_n}$$

with  $\text{diam } X_{i_1, \dots, i_n} \leq d_0/2^n$  pick two different points  $x_{i_1, \dots, i_n, 0}$  and  $x_{i_1, \dots, i_n, 1}$  in  $\text{Int } X_{i_1, \dots, i_n}$  and let  $Y_{i_1, \dots, i_n, i_{n+1}} := \overline{B(x_{i_1, \dots, i_n, i_{n+1}}, d(x_{i_1, \dots, i_n, 0}, x_{i_1, \dots, i_n, 1})/4)}$ ,  $X_{i_1, \dots, i_n, i_{n+1}} := Y_{i_1, \dots, i_n, i_{n+1}} \cap C_n$  and

$$C_{n+1} = \bigcup_{i_1, \dots, i_n, i_{n+1} \in \{0,1\}} X_{i_1, \dots, i_n, i_{n+1}}.$$

Since  $\text{diam } X_{i_1, \dots, i_n} \leq d_0/2^n$ , each infinite intersection

$$\bigcap_{i_1, \dots, i_n, \dots \in \{0,1\}} X_{i_1, \dots, i_n, \dots}$$

is a single point by Proposition 2.9. The set  $C := \bigcap_{n=1}^{\infty} C_n$  is homeomorphic to the countable product of the two point sets  $\{0, 1\}$  via the map

$$\bigcap_{i_1, \dots, i_n, \dots \in \{0,1\}} X_{i_1, \dots, i_n, \dots} \mapsto (i_1, \dots, i_n, \dots).$$

By Example 1.41  $C$  is homeomorphic to the Cantor set. □

The theorem is thus proved. □

**e. Completion.** Completeness is an important property since it allows us to perform limit operations which arise frequently in our constructions. Notice that it is not possible to define a notion of Cauchy sequences in an arbitrary topological space since one lacks the possibility of comparing neighborhoods at different points. Here the uniform structure provides the most general natural setting. A metric space can be made complete in the following way:

**DEFINITION 2.14.** If  $X$  is a metric space and there is an isometry from  $X$  onto a dense subset of a complete metric space  $\hat{X}$  then  $\hat{X}$  is called the *completion* of  $X$ .

**THEOREM 2.15.** *For any metric space  $X$  there exists a completion unique up to an isometry which commutes with the embeddings of  $X$  into a completion.*

**PROOF.** The process mimics the well-known from basic real analysis construction of real numbers as the completion of rationals. Namely the elements of the completion are equivalence classes of Cauchy sequences by identifying two sequences if the distance between corresponding elements converges to zero. The distance between two (equivalence

classes of) sequences is defined as the limit of the distance between corresponding elements. The isometric embedding of  $X$  into the completion is given by identifying element of  $X$  with constant sequences. Uniqueness is obvious by definition since by uniform continuity the isometric embedding of  $X$  to any completion extends to an isometric bijection of the standard completion.  $\square$

**f.  $p$ -adic completions of integers and rationals.** This is an example which rivals the construction of real numbers in its importance for various areas of mathematics, especially to the number theory and algebraic geometry. Unlike the construction of the reals it gives infinitely many different nonisometric completions of the rationals.

Let  $p$  be a positive prime number. Any rational number  $r$  can be represented as  $p^m \frac{k}{l}$  where  $m$  is an integer and  $k$  and  $l$  are integers relatively prime with  $p$ . Define the  $p$ -adic norm  $\|r\|_p := p^{-m}$  and the distance  $d_p(r_1, r_2) := \|r_1 - r_2\|_p$ .

EXERCISE 0.7. Show that  $p$ -adic norm is *multiplicative*:  $\|r_1 \cdot r_2\|_p = \|r_1\|_p \|r_2\|_p$ .

PROPOSITION 2.16. For  $r_1, r_2, r_3 \in \mathbb{Q}$ ,  $d_p(r_1, r_3) \leq \max(d_p(r_1, r_2), d_p(r_2, r_3))$

REMARK 2.17. A metric satisfying this property (which is stronger than the triangle inequality) is called an *ultrametric*.

PROOF. Since  $\|r\|_p = \|-r\|_p$  the statement follows from the property of  $p$ -norms:

$$\|r_1 + r_2\|_p \leq \|r_1\|_p + \|r_2\|_p.$$

To see this write  $r_i = p_i^{m_i} \frac{k_i}{l_i}$ ,  $i = 1, 2$  with  $k_i$  and  $l_i$  relatively prime with  $p$  and assume without loss of generality that  $m_2 \geq m_1$ . We have

$$r_1 + r_2 = p_1^{m_1} \frac{k_1 l_2 + p^{m_2 - m_1} k_2 l_1}{l_1 l_2}.$$

The numerator  $k_1 l_2 + p^{m_2 - m_1} k_2 l_1$  is an integer and if  $m_2 > m_1$  it is relatively prime with  $p$ . In any event we have  $\|r_1 + r_2\|_p \leq p^{-m_1} = \|r_1\|_p = \max(\|r_1\|_p, \|r_2\|_p)$ .  $\square$

Proposition 2.16 and the multiplicativity property of the  $p$ -adic norm allow to extend correspondingly the addition and the multiplication from  $\mathbb{Q}$  to the completion. This is done in exactly the same way as in the real analysis for real numbers. Existence of the opposite and inverse (the latter for a nonzero element) follow. Thus the completion becomes a field which is called the *field of  $p$ -adic numbers* and is usually denoted by  $\mathbb{Q}_p$ . Restricting the procedure to the integers which always have norm  $\leq 1$  one obtains the subring of  $\mathbb{Q}_p$  which is called the *ring of  $p$ -adic integers* and is usually denoted by  $\mathbb{Z}_p$ .

The topology of  $p$ -adic numbers is once again indicates the importance of the Cantor set.

PROPOSITION 2.18. The space  $\mathbb{Z}_p$  is homeomorphic to the Cantor set;  $\mathbb{Z}_p$  is the unit ball (both closed and open) in  $\mathbb{Q}_p$ .

The space  $\mathbb{Q}_p$  is homeomorphic to the disjoint countable union of Cantor sets.

PROOF. We begin with the integers. For any sequence  $a = \{a_n\} \in \prod_{n=1}^{\infty} \{0, 1, \dots, p-1\}$  the sequence of integers

$$k_n(a) := \sum_{i=1}^n a_i p^i$$

is Cauchy; for different  $\{a_n\}$  these sequences are non equivalent and any Cauchy sequence is equivalent to one of these. Thus the correspondence  $\prod_{n=1}^{\infty} \{0, 1, \dots, p-1\} \rightarrow \mathbb{Z}_p$ , given by

$$\{a_n\} \mapsto \text{the equivalence class of } k_n(a)$$

is a homeomorphism. The space  $\prod_{n=1}^{\infty} \{0, 1, \dots, p-1\}$  can be mapped homeomorphically to a nowhere dense perfect subset of the interval by the map

$$\{a_n\}_{n=1}^{\infty} \mapsto \sum_{n=1}^{\infty} a_n (2p-1)^{-i}$$

. Thus the statement about  $\mathbb{Z}_p$  follows from Proposition 1.43.

Since  $\mathbb{Z}$  is the unit ball (open and closed) around 0 in the metric  $d_p$  and any other point is at a distance atleast 1 from it, the same holds for the completions.

Finally any rational number can be uniquely represented as  $k + \sum_{i=1}^n a_i p^{-i}$  where  $k$  is an integer and  $a_i \in \{0, \dots, p-1\}$ ,  $i = 1, \dots, n$ . If the corresponding finite sequences  $a_i$  have different length or do not coincide the  $p$ -adic distance between the rationals is at least 1. Passing to the completion we see that any  $x \in \mathbb{Q}_p$  is uniquely represented as  $k + \sum_{i=1}^{\infty} a_i p^{-i}$  with  $k \in \mathbb{Z}_p$ . with pairwise distances for different  $a_i$ 's at least one.  $\square$

### g. Separation properties.

PROPOSITION 2.19. *Any metric space is normal as a topological space.*

PROOF. For two disjoint closed sets  $A, B \in X$  let  $\mathcal{O}_A := \{x \in X \mid d(x, A) < d(x, B)\}$  and  $\mathcal{O}_B := \{x \in X \mid d(x, B) < d(x, A)\}$ . this sets are open disjoint and contain  $A$  and  $B$  correspondingly.  $\square$

A topological space is said to be *metrizable* if there exists a metric on it that generates the given topology. The following *Urysohn Metrization Theorem* gives necessary and sufficient conditions for metrizability for second countable topological spaces. It follows from Proposition 1.26 and Theorem 9.10 from Bredon.

THEOREM 2.20. *A normal space with a countable base for the topology, hence any compact Hausdorff space with a countable base, is metrizable.*

### h. Compact metric spaces.

#### 1. Sequential compactness.

PROPOSITION 2.21. *Any compact metric space is complete.*

PROOF. Suppose the opposite, that is,  $X$  is a compact metric space and a Cauchy sequence  $x_n$ ,  $n = 1, 2, \dots$  does not converge. By taking a subsequence if necessary we may assume that all points  $x_n$  are different. The union of the elements of the sequence is closed since the sequence does not converge. Let

$$\mathcal{O}_n := X \setminus \bigcup_{i=n}^{\infty} \{x_i\}$$

These sets form an open cover of  $X$  but since they are increasing there is no finite subcover.  $\square$

For metric spaces there is a familiar characterization of compactness.

**PROPOSITION 2.22.** *A metric space is compact if and only if every sequence has a converging subsequence.*

**PROOF.** Suppose  $f$  is compact. By Proposition 2.21 it is sufficient to show that every sequence has a Cauchy subsequence. Take a sequence  $x_n$ ,  $n = 1, 2, \dots$  and consider the cover by all balls of radius 1. Since it has a finite subcover there is a ball of radius 1 which contains infinitely many elements of the sequence. Consider only these elements as a subsequence. Take cover by all balls of radius  $1/2$ , choose a finite subcover and a subsequence which lies in a single ball of radius  $1/2$ . Continuing by induction we find nested subsequences of the original sequence which lie in balls of radius  $1/2^n$ . Using the standard diagonal process we construct a Cauchy subsequence.

To prove the reverse implication let us first show that the space must be separable. This implies that any cover contains a countable subcover since the space has countable base. If the space is not separable than there exists an  $\epsilon > 0$  such that for any countable (and hence finite) collection of points there is a point at the distance greater than  $\epsilon$  from all of them. This allows to construct by induction a sequence of points which are pairwise more than  $\epsilon$  apart. Such a sequence obviously does not contain a converging subsequence.

Now assume there is an open countable cover  $\{\mathcal{O}_1, \mathcal{O}_2, \dots\}$  without a finite subcover. Take the union of the first  $n$  elements of the cover and a point  $x_n$  outside of the union. The sequence  $x_n$ ,  $n = 1, 2, \dots$  thus defined has a converging subsequence  $x_{n_k} \rightarrow x$ . But  $x$  belong to a certain element of the cover, say  $\mathcal{O}_N$ . Then for a sufficiently large  $k$ ,  $n_k > N$  hence  $x_{n_k} \notin \mathcal{O}_N$ , a contradiction to convergence.  $\square$

## 2. Lebesgue number.

**PROPOSITION 2.23.** *For an open cover of a compact metric space there exists a number  $\delta$  such that every  $\delta$ -ball is contained in an element of the cover.*

**PROOF.** Suppose the opposite. Then there exists a cover and a sequence of points  $x_n$  such that the ball  $B(x_n, 1/2^n)$  does not belong to any element of the cover. Take a converging subsequence  $x_{n_k} \rightarrow x$ . Since the point  $x$  is covered by an open set, a ball of radius  $r > 0$  around  $x$  belongs to that element. But for  $k$  large enough  $d(x, x_{n_k}) < r/2$  and hence by the triangle inequality the ball  $B(x_{n_k}, r/2)$  lies in the same element of the cover.  $\square$

The largest such number is called the *Lebesgue number* of the cover.

## 3. Characterization of Cantor sets.

**THEOREM 2.24.** *Any perfect compact totally disconnected metric space is homeomorphic to the Cantor set.*

**PROOF.** In a totally disconnected metric space any point is contained inside a set of arbitrarily small diameter which is both closed and open. (see the next exercise) So consider a cover of the space by such sets of diameter  $\leq 1$ . Take a finite subcover. Since any finite intersection of such sets is still both closed and open by taking all possible intersection we obtain a *partition* of the space into finitely many closed and open sets of diameter  $\leq 1$ . Since the space is perfect no element of this partition is a point so a further division is possible. Repeating this procedure for each set in the cover by covering it by sets of

diameter  $\leq 1/2$  we obtain a finer partition into closed and open sets of diameter  $\leq 1/2$ . Proceeding by induction we obtain a nested sequence of finite partitions into closed and open sets of positive diameter  $\leq 1/2^n$ ,  $n = 0, 1, 2, \dots$ . Proceeding as in the proof of Proposition 1.43, that is, mapping elements of each partition inside a nested sequence of contracting intervals, we construct a homeomorphism of the space onto a nowhere dense perfect subset of  $[0, 1]$  and hence by Proposition 1.43 our space is homeomorphic to the Cantor set.  $\square$

EXERCISE 0.8. Prove that in a totally disconnected metric space any point is contained inside a set of arbitrarily small diameter which is both closed and open.

4. *Universality of the Hilbert cube.*

THEOREM 2.25. *Any compact separable metric space  $X$  is homeomorphic to a closed subset of the Hilbert cube  $H$ .*

PROOF. First by multiplying the metric by a constant if necessary we may assume that the diameter of  $X$  is less than 1. Pick a dense sequence of points  $x_1, x_2, \dots$  in  $X$ . Let  $F : X \rightarrow H$  be defined by

$$F(x) = (d(x, x_1), d(x, x_2), \dots).$$

This map is injective since for any two distinct points  $x$  and  $x'$  one can find  $n$  such that  $d(x, x_n) < (1/2)d(x', x_n)$  so that by the triangle inequality  $d(x, x_n) < d(x', x_n)$  and hence  $F(x) \neq F(x')$ . By Proposition 1.30  $F(X) \subset H$  is compact and by Proposition 1.31  $F$  is a homeomorphism between  $X$  and  $F(X)$ .  $\square$

EXERCISE 0.9. Prove the infinite-dimensional torus  $\mathbb{T}^\infty$ , the product of the countably many copies of the unit circle, has the same universality property as the Hilbert cube, that is, any compact separable metric space  $X$  is homeomorphic to a closed subset of  $\mathbb{T}^\infty$ .

**i. Spaces of continuous maps.** If  $X$  is a compact metrizable topological space (for example, a compact manifold), then the space  $C(X, X)$  of continuous maps of  $X$  into itself possesses the  $C^0$  or *uniform* topology. It arises by fixing a metric  $\rho$  in  $X$  and defining the distance  $d$  between  $f, g \in C(X, X)$  by

$$d(f, g) := \max_{x \in X} \rho(f(x), g(x)).$$

The subset  $\text{Hom}(X)$  of  $C(X, X)$  of homeomorphisms of  $X$  is neither open nor closed in the  $C^0$  topology. It possesses, however, a natural topology as a complete metric space induced by the metric

$$d_H(f, g) := \max(d(f, g), d(f^{-1}, g^{-1})).$$

If  $X$  is  $\sigma$ -compact we introduce the compact–open topologies for maps and homeomorphisms, that is, the topologies of uniform convergence on compact sets.

We sometimes use the fact that equicontinuity gives some compactness of a family of continuous functions in the uniform topology.

THEOREM 2.26 (Arzelà–Ascoli Theorem). *Let  $X, Y$  be metric spaces,  $X$  separable, and  $\mathcal{F}$  an equicontinuous family of maps. If  $\{f_i\}_{i \in \mathbb{N}} \subset \mathcal{F}$  such that  $\{f_i(x)\}_{i \in \mathbb{N}}$  has compact closure for every  $x \in X$  then there is a subsequence converging uniformly on compact sets to a function  $f$ .*

Thus in particular a closed bounded equicontinuous family of maps on a compact space is compact in the uniform topology (induced by the maximum norm).

Let us sketch the proof. First use the fact that  $\{f_i(x)\}_{i \in \mathbb{N}}$  has compact closure for every point  $x$  of a countable dense subset  $S$  of  $X$ . A diagonal argument shows that there is a subsequence  $f_{i_k}$  which converges at every point of  $S$ . Now equicontinuity can be used to show that for every point  $x \in X$  the sequence  $f_{i_k}(x)$  is Cauchy, hence convergent (since  $\{f_i(x)\}_{i \in \mathbb{N}}$  has compact, hence complete, closure). Using equicontinuity again yields continuity of the pointwise limit. Finally a pointwise convergent equicontinuous sequence converges uniformly on compact sets.

### 3. Elementary homotopy theory

#### a. Homotopy and homotopy equivalence.

DEFINITION 3.1. Two continuous maps  $h_0, h_1: X \rightarrow Y$  between topological spaces are said to be *homotopic* if there exists a continuous map  $h: [0, 1] \times X \rightarrow Y$  (the *homotopy*) such that  $h(i, \cdot) = h_i$  for  $i = 0, 1$ . If  $h_0(x) = h_1(x) = p$  for some  $x \in X$  then  $h_0, h_1: X \rightarrow Y$  are called *homotopic rel  $p$*  if  $h$  can be chosen such that  $h(\cdot, x) = p$ . If  $X = [0, 1]$ ,  $h_0(0) = h_1(0)$ , and  $h_0(1) = h_1(1)$  then we say that  $h_0, h_1$  are *homotopic rel endpoints* if  $h(\cdot, 0)$  and  $h(\cdot, 1)$  can be taken constant.  $h$  is called *null-homotopic* if it is homotopic to a constant map. If  $h_1, h_2$  are homeomorphisms they are called *isotopic* if  $h$  can be taken such that every  $h(t, \cdot)$  is a homeomorphism.

Topological spaces  $X$  and  $Y$  are called *homotopically equivalent* if there exist maps  $g: X \rightarrow Y$  and  $h: Y \rightarrow X$  such that  $g \circ h$  and  $h \circ g$  are homotopic to the identity.

A property of a topological space which is the same for any homotopically equivalent space is called a *homotopy invariant*.

Obviously homeomorphic spaces are homotopically equivalent. Thus homotopy invariants represent a class of topological invariants some of which are amenable to more or less straightforward calculations.

#### b. Contractible spaces.

DEFINITION 3.2. A topological space  $X$  is called *contractible* if it is homotopically equivalent to a point.

Equivalently a space is contractible if the identity map is null-homotopic.

Contractible spaces are the simplest from the point of view of homotopy invariance.

PROPOSITION 3.3. *Any contractible space is path connected.*

PROOF. Let  $x_1, x_2 \in X$  where  $X$  is contractible. Take a homotopy  $h$  between the identity and a constant map, to, say  $x_0$ . Let

$$f(t) := \begin{cases} h(x, 2t) & \text{when } t \leq \frac{1}{2}, \\ h(y, 2t - 1) & \text{when } t \geq \frac{1}{2}. \end{cases}$$

Thus  $f$  is a continuous map of  $[0, 1]$  to  $X$  with  $f(0) = x$  and  $f(1) = y$ .  $\square$

PROPOSITION 3.4. *Any convex subset of  $\mathbb{R}^n$  is contractible.*

PROOF. Let  $C$  be a convex set in  $\mathbb{R}^n$  and let  $x_0 \in C$ . Define

$$h(x, t) = x_0 + (1 - t)(x - x_0).$$

By convexity for any  $t \in [0, 1]$  we obtain a map of  $C$  into itself. This is a homotopy between the identity and the constant map to  $x_0$   $\square$

PROPOSITION 3.5. *If  $X$  is contractible then for any topological space  $Y$  the product  $X \times Y$  is homotopically equivalent to  $X$ .*

PROOF. If  $h: Y \times [0, 1] \rightarrow Y$  is a homotopy between the identity and a constant map of  $Y$ , that is,  $h(y, 0) = y$  and  $h(y, 1) = y_0$ . Then for the map  $H := \text{Id}_X \times h$  one has  $H(x, y, 0) = (x, y)$  and  $H(x, y, 1) = (x, y_0)$ . Thus the projection  $\pi_1: (x, y) \mapsto x$  and the embedding  $i_{y_0}: x \mapsto (x, y_0)$  provide a homotopy equivalence.  $\square$

EXAMPLE 3.6. The circle  $S^1$  and the cylinder  $S^1 \times \mathbb{R}$  are homotopically equivalent but not homeomorphic.

**c. Degree of a circle map.** Contractible spaces are trivial from the homotopy equivalence point of view. Now we will discuss the simplest nontrivial case where homotopy invariants can be calculated. Recall the relation between the circle  $S^1 = \mathbb{R}/\mathbb{Z}$  and the line  $\mathbb{R}$ . There is a projection  $\pi: \mathbb{R} \rightarrow S^1$ ,  $x \mapsto [x]$ , where  $[x]$  is the equivalence class of  $x$  in  $\mathbb{R}/\mathbb{Z}$ . Recall that  $[\cdot]$  denotes an equivalence class, whereas the integer part of a number is written  $\lfloor \cdot \rfloor$ . We use  $\{\cdot\}$  for the fractional part.

PROPOSITION 3.7. *If  $f: S^1 \rightarrow S^1$  is continuous then there exists a continuous  $F: \mathbb{R} \rightarrow \mathbb{R}$ , called a lift of  $f$  to  $\mathbb{R}$ , such that*

$$(3.1) \quad f \circ \pi = \pi \circ F,$$

that is,  $f([z]) = [F(z)]$ . Such a lift is unique up to an additive integer constant and  $\deg(f) := F(x+1) - F(x)$  is an integer independent of  $x \in \mathbb{R}$  and the lift  $F$ . It is called the degree of  $f$ . If  $f$  is a homeomorphism then  $|\deg(f)| = 1$ .

PROOF. Existence: Pick a point  $p \in S^1$ . Then  $p = [x_0]$  for some  $x_0 \in \mathbb{R}$  and  $f(p) = [y_0]$  for some  $y_0 \in \mathbb{R}$ . From these choices of  $x_0$  and  $y_0$  define  $F: \mathbb{R} \rightarrow \mathbb{R}$  by requiring that  $F(x_0) = y_0$ ,  $F$  is continuous, and  $f([z]) = [F(z)]$  for all  $z \in \mathbb{R}$ . One can construct such an  $F$  by varying the initial point  $p$  continuously, which causes  $f(p)$  to vary continuously. Then there is no ambiguity of how to vary  $x$  and  $y$  continuously and thus  $F(x) = y$  defines a continuous map.

To elaborate, take  $\delta > 0$  such that  $d([x], [x']) \leq \delta$  implies  $d(f([x]), f([x'])) < 1/2$ . Then we can define  $F$  on  $[x_0 - \delta, x_0 + \delta]$  as follows: If  $|x - x_0| \leq \delta$  then  $d(f([x]), q) < 1/2$  and there is a unique  $y \in (y_0 - 1/2, y_0 + 1/2)$  such that  $[y] = f([x])$ . Define  $F(x) = y$ . Analogous steps extend the domain by another  $\delta$  at a time, until  $F$  is defined on an interval of unit length. (One needs to check some consistency.) Then  $f([z]) = [F(z)]$  defines  $F$  on  $\mathbb{R}$ .

Uniqueness: Suppose  $\tilde{F}$  is another lift. Then  $[\tilde{F}(x)] = f([x]) = [F(x)]$  for all  $x$ , meaning  $\tilde{F} - F$  is always an integer. Because it is continuous it must be constant.

Degree:  $F(x+1) - F(x)$  is an integer (now evidently independent of the choice of lift) because  $[F(x+1)] = f([x+1]) = f([x]) = [F(x)]$ . By continuity  $F(x+1) - F(x) =: \deg(f)$  must be a constant.

Invertibility: If  $\deg(f) = 0$  then  $F(x+1) = F(x)$  and thus  $F$  is not monotone. Then  $f$  is noninvertible because it cannot be monotone. If  $|\deg(f)| > 1$  then  $|F(x+1) - F(x)| > 1$  and by the Intermediate Value Theorem there exists a  $y \in (x, x+1)$  with  $|F(y) - F(x)| = 1$ , hence  $f([y]) = f([x])$ , and  $[y] \neq [x]$ , so  $f$  is noninvertible.  $\square$

PROPOSITION 3.8. *Degree is a homotopy invariant.*

PROOF. The lift construction can be simultaneously applied to a continuous family of circle maps to produce a continuous family of lifts. Hence degree have to change continuously under homotopy. Since it is an integer it is in fact constant.  $\square$

COROLLARY 3.9. *The circle is not contractible.*

PROOF. Degrees of any constant map is equal to zero and for the identity map it is equal to one.  $\square$

**THEOREM 3.10.** *Degree is a complete homotopy invariant of circle self-maps: for any  $m \in \mathbb{Z}$  any map of degree  $m$  is homotopic to the map*

$$E_m := x \mapsto mx \pmod{1}.$$

**PROOF.** Obviously the map  $E_m$  lifts to the linear map  $x \mapsto mx$  of  $\mathbb{R}$ . On the other hand every lift  $F$  of a degree  $m$  map  $f$  has the form  $F(x) = mx + H(x)$ , where  $H$  is a periodic function with period one. Thus the family of maps

$$F_t(x) := mx + (1-t)H(x)$$

are lifts of a continuous family of maps of  $S^1$  which provide a homotopy between  $f$  and  $E_m$ .  $\square$

**EXERCISE 0.1.** Any continuous map  $f : S^1 \rightarrow S^1$  has at least  $|\deg f - 1|$  fixed points.

**d. Brouwer fixed point theorem.** While degree theory for circle maps is quite elementary it provides the tools for proving one of the most celebrated classical results in topology.

**THEOREM 3.11 (Brouwer fixed-point theorem in dimension two).** *Any continuous map of a closed disc into itself (and hence of any space homeomorphic to the disc) has a fixed point.*

**PROOF.** We consider the standard closed disc

$$\mathbb{D}^2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Suppose  $f : \mathbb{D}^2 \rightarrow \mathbb{D}^2$  is a continuous map without fixed points. For  $p \in \mathbb{D}^2$  consider the open halfline beginning at  $F(p)$  and passing through the point  $p$ . This halfline intersects the unit circle  $S^1$ , which is the boundary of the disc  $\mathbb{D}^2$ , at a single point which we will denote by  $h(p)$ . Notice that for  $p \in \partial\mathbb{D}^2$ ,  $h(p) = p$ . The map  $h : \mathbb{D}^2 \rightarrow \partial\mathbb{D}^2$  thus defined is continuous by construction and is homotopic to the identity map  $\text{Id}_{\mathbb{D}^2}$  via the straight-line homotopy  $H(p, t) = (1-t)p + th(p)$ . Now identify  $\partial\mathbb{D}^2$  with the unit circle  $S^1$ ; Taking the composition of  $h$  with this identification we obtain a map  $\mathbb{D}^2 \rightarrow S^1$  which we will denote by  $g$ . Let  $i : S^1 \rightarrow \mathbb{D}^2$  be the standard embedding. We have

$$g \circ i : S^1 \rightarrow S^1 = \text{Id}_{S^1}, \quad i \circ g = h \quad \text{is homotopic to } \text{Id}_{\mathbb{D}^2}.$$

Thus the pair  $(i, g)$  gives a homotopy equivalence between  $S^1$  and  $\mathbb{D}^2$ . But this is impossible since the disc is contractible and the circle is not (Corollary 3.9). Hence such a map  $h$  cannot be constructed; this implies that  $F$  has a fixed point at which the halfline in question cannot be uniquely defined.  $\square$

**EXERCISE 0.2.** Any continuous map  $f : S^2 \rightarrow S^2$  has a fixed point or a point which is mapped to the diametrically opposite point

**EXERCISE 0.3.** Any continuous map  $f : \mathbb{R}P(2) \rightarrow \mathbb{R}P(2)$  of the projective plane into itself has a fixed point.

**EXERCISE 0.4.** Deduce the general form of the Brouwer fixed-point theorem: *Any continuous map of a closed  $n$ -disc into itself has a fixed point,* from the fact that the identity map on the sphere of any dimension is not null-homotopic.

**e. The fundamental group.** Let  $M$  be a topological space  $p \in M$ , and consider the collection of curves  $c: [0, 1] \rightarrow M$  with  $c(0) = c(1) = p$ . If  $c_1$  and  $c_2$  are such curves then let  $c_1 \cdot c_2$  be the curve given by

$$c_1 \cdot c_2(t) := \begin{cases} c_1(2t) & \text{if } t \leq \frac{1}{2}, \\ c_2(2t - 1) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

**PROPOSITION 3.12.** *Classes of curves as above homotopic rel endpoints form a group with respect to the operation induced by  $\cdot$ .*

**PROOF.** First notice that the operation is indeed defined on the homotopy classes. For, if the paths  $c_i$  are homotopic to  $\bar{c}_i$ ,  $i = 1, 2$  via the maps  $h_i: [0, 1] \times [0, 1] \rightarrow M$  then the map  $h$ , defined by

$$h(t, s) := \begin{cases} h_1(2t, s) & \text{if } t \leq \frac{1}{2}, \\ h_2(2t - 1, s) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Obviously the role of the unit is played by the homotopy class of the constant map  $c_0(t) = p$ . Then the inverse to  $c$  will be the homotopy class of the map  $c'(t) := c(1 - t)$ . What remains is to check the associative law:  $(c_1 \cdot c_2) \cdot c_3$  is homotopic rel  $p$  to  $c_1 \cdot (c_2 \cdot c_3)$  and to show that  $c \cdot c'$  is homotopic to  $c_0$ . In both cases the homotopy is done by reparametrization in the preimage, that is, on the square  $[0, 1] \times [0, 1]$ .

For the associativity consider the following continuous map (“reparametrization”) of the square into itself

$$R(t, s) = \begin{cases} (t(1+s), s) & \text{if } 0 \leq t \leq \frac{1}{4}, \\ (t + \frac{s}{4}, s) & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2}, \\ (1 - \frac{1}{1+s} + \frac{t}{1+s}, s) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then the map  $c_1 \cdot (c_2 \cdot c_3) \circ R: [0, 1] \times [0, 1] \rightarrow M$  provides a homotopy rel  $p$  between the curves  $c_1 \cdot (c_2 \cdot c_3)$  and  $(c_1 \cdot c_2) \cdot c_3$ .

Similarly, a homotopy between  $c \cdot c'$  and  $c_0$  is given by  $c \cdot c' \circ I$  where the reparametrization  $I: [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$  is defined as

$$I(t, s) = \begin{cases} (t, s) & \text{if } 0 \leq t \leq \frac{1-s}{2}, \text{ or } \frac{1+s}{2} \leq t \leq 1, \\ (\frac{1-s}{2}, s) & \text{if } \frac{1-s}{2} \leq t \leq \frac{1+s}{2}, \end{cases}$$

Notice that while the reparametrization  $I$  is discontinuous along the wedge  $t = (1 \pm s)/2$  the map  $(c \cdot c') \circ I$  is continuous by the definition of  $c'$ .  $\square$

**DEFINITION 3.13.** The group described in Proposition 3.12 is called the *fundamental group*  $\pi_1(M, p)$  of  $M$  at  $p$ .

**PROPOSITION 3.14.** *If  $p$  and  $q$  belong to the same path connected component of  $M$  then the groups  $\pi_1(M, p)$  and  $\pi_1(M, q)$  are isomorphic.*

**PROOF.** Let  $\rho: [0, 1] \rightarrow M$  be a path connecting points  $p$  and  $q$ . It is natural to denote the path  $\rho \circ S$  where  $S(t) = 1 - t$  by  $\rho^{-1}$ . It is also natural to extend the “ $\cdot$ ” operation to path with different endpoints if they match properly. With these conventions established we associate to a path  $c: [0, 1] \rightarrow M$  with  $c(0) = c(1) = p$  the path  $c' := \rho^{-1} \cdot c \cdot \rho$  with  $c'(0) = c'(1) = q$ . In order to finish the proof one needs to show that this correspondence takes paths homotopic rel  $p$  to paths homotopic rel  $q$ , respects the group operation and

is bijective up to homotopy. These statements are proved using appropriate rather natural reparametrizations similarly to the proof of Proposition 3.12.  $\square$

Furthermore, it follows from the construction that different choices of the connecting path  $\rho$  will produce isomorphisms between  $\pi_1(M, p)$  and  $\pi_1(M, q)$  which differ by an inner automorphism of either group.

If in particular the space  $M$  is path connected (in the manifold case connectedness is sufficient) then fundamental groups at all of its points are isomorphic and one talks simple about the *fundamental group of  $M$*  which is sometime denoted by  $\pi_1(M)$ .

A path connected space with trivial fundamental group is said to be *simply connected*, or sometimes *1-connected*.

Since the fundamental group is defined modulo homotopy, it is the same for homotopically equivalent spaces, that is, it is a homotopy invariant. The *free* homotopy classes of curves (that is, with no fixed base point) correspond exactly to the conjugacy classes of curves modulo changing base point, so there is a natural bijection between the classes of freely homotopic closed curves and conjugacy classes in the fundamental group.

Fundamental group behaves nicely with respect to some basic constructions.

**PROPOSITION 3.15.** *If  $X$  and  $Y$  are path connected spaces then*

$$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y).$$

#### f. Covering spaces.

**DEFINITION 3.16.** If  $M, M'$  are topological manifolds and  $\pi: M' \rightarrow M$  is a continuous map such that  $\text{card } \pi^{-1}(y)$  is independent of  $y \in M$  and every  $x \in \pi^{-1}(y)$  has a neighborhood on which  $\pi$  is a homeomorphism to a neighborhood of  $y \in M$  then  $\pi$  is called a *covering map* and  $M'$  (or  $(M', \pi)$ ) is called a *covering (space) or cover* of  $M$ . If  $n = \text{card } \pi^{-1}(y)$  is finite then  $(M', \pi)$  is said to be an  *$n$ -fold covering*. If  $f: N \rightarrow M$  is continuous and  $F: N \rightarrow M'$  is such that  $f = \pi \circ F$  then  $F$  is said to be a *lift* of  $f$ . If  $f: M \rightarrow M$  is continuous and  $F: M' \rightarrow M'$  is continuous such that  $f \circ \pi = \pi \circ F$  then  $F$  is said to be a *lift* of  $f$  as well. A simply connected covering is called the *universal cover*. A homeomorphism of a covering  $M'$  of  $M$  is called a *deck transformation* if it is a lift of the identity on  $M$ .

**EXAMPLE 3.17.**  $(\mathbb{R}, \exp(2\pi i(\cdot)))$  is a covering of the unit circle. Geometrically one can view this as the helix  $(e^{2\pi ix}, x)$  covering the unit circle under projection. The map defined by taking the fractional part likewise defines a covering of the circle  $\mathbb{R}/\mathbb{Z}$  by  $\mathbb{R}$ .

**PROPOSITION 3.18.** *If  $\pi: M' \rightarrow M$  and  $\rho: N' \rightarrow N$  are covering maps then  $\pi \times \rho: M' \times N' \rightarrow M \times N$  is a covering map*

**EXAMPLE 3.19.** The torus  $\mathbb{T}^2 = S^1 \times S^1$  is covered by the cylinder  $S^1 \times \mathbb{R}$  which is in turn covered by  $\mathbb{R}^2$ . Notice that the fundamental group  $\mathbb{Z}$  of the cylinder is a subgroup of that of the torus ( $\mathbb{Z}^2$ ) and  $\mathbb{R}^2$  is a simply connected cover of both.

**EXAMPLE 3.20.** The maps  $E_m, |m| \geq 2$  of the circle define coverings of the circle by itself.

See Bredon, Chapter 3, in particular, Section 3.8 for a detailed discussion. For the time being we give just a short summary.

As it turns out the structure of all covering spaces of a given connected manifold  $M$  can be completely described algebraically through the fundamental group. There is a natural bijection between conjugacy classes of subgroups of  $\pi_1(M)$  and classes of covering spaces modulo homeomorphisms commuting with deck transformations. In particular the universal cover is unique. This bijection can be described as follows. Suppose  $(M', \pi)$  is a covering of  $M$  and  $x_0, x_1 \in \pi^{-1}(y)$ . Since  $M'$  is path connected there are curves  $c: [0, 1] \rightarrow M'$  with  $c(i) = x_i$  for  $i = 0, 1$ . Under  $\pi$  these project to loops on  $M$ . Any continuous map induces a homomorphism between the fundamental groups. Any continuous map possesses a lift, so a homotopy of the loop  $\pi \circ c$  rel  $y$  can be lifted to a homotopy of the curve  $c$  and since by hypothesis  $\pi^{-1}(y)$  is discrete, this homotopy is a homotopy rel endpoints. In particular homotopic curves project to homotopic curves and, by considering the case  $x_0 = x_1$ , the fundamental group of  $M'$  injects into the fundamental group of  $M$  as a subgroup. This is the subgroup corresponding to the covering. Furthermore this subgroup is a proper subgroup whenever  $\pi$  is not a homeomorphism, that is, the cover is a nontrivial covering. Thus a simply connected space has no proper coverings. One can also see that any two coverings  $M'_1$  and  $M'_2$  of  $M$  have a common covering  $M''$ , so the universal cover is unique. Any topological manifold has a universal cover.

## 4. Real and complex differentiable manifolds; A first introduction

### a. Real differentiable manifolds.

DEFINITION 4.1. A  $n$ -dimensional topological manifold  $M$  is called an  $n$ -dimensional *differentiable manifold* if it is covered by a family  $\mathfrak{A} = \{(U_i, h_i)\}_i$  of charts such that for any two charts  $(U_1, h_1)$  and  $(U_2, h_2)$  in  $\mathfrak{A}$  with  $h_i: U_i \rightarrow B_i \subset \mathbb{R}^n$  the *coordinate change*  $h_2 \circ h_1^{-1}$  is differentiable on  $h_1(U_1 \cap U_2) \subset B_1$ . Here differentiable can be taken to mean  $C^r$  for any  $r \in \mathbb{N} \cup \infty$ , or analytic. A collection of such charts covering  $M$  is called an *atlas* of  $M$ . Any atlas defines a unique *maximal atlas* by taking all charts compatible with the present ones. A maximal atlas is called a *differentiable structure*.

One should note that the differentiable structure is obtained in a different way from the topological structure: The latter is given a priori, whereas a differentiable structure is obtained from  $\mathbb{R}^n$  via a compatibility condition on charts.

#### 1. Examples.

EXAMPLE 4.2.  $\mathbb{R}^n$  is a smooth manifold with the identity as a chart, as are its open subsets.

An interesting example of this kind is obtained by viewing the linear space of  $n \times n$  matrices as  $\mathbb{R}^{n^2}$ . The condition  $\det A \neq 0$  then defines an open set, hence a manifold, which is familiar as the *general linear group*  $GL(n, \mathbb{R})$  of invertible  $n \times n$  matrices.

Simple smooth curves and surfaces in  $\mathbb{R}^n$  are manifolds: Any local piece of the parameterization gives a chart (its inverse).

EXAMPLE 4.3. The standard sphere  $S^2 \subset \mathbb{R}^3$  is a manifold (as charts one can take the six parallel projections of hemispheres to coordinate planes or the stereographic projections of the sphere minus a pole).

EXAMPLE 4.4. The embedded torus (doughnut) is a manifold via any pieces of the parameterization.

Note that also nonsmooth curves can be viewed as smooth manifolds, for example, a simple curve with a corner (like “└”) is homeomorphic to  $\mathbb{R}$ , so this single global chart defines a differentiable structure. Of course this structure is incompatible with the ambient one so this example is not a smooth submanifold of  $\mathbb{R}^n$ .

2. *Manifolds defined by equations.* Joint level sets of smooth functions into  $\mathbb{R}$  or  $\mathbb{R}^m$  corresponding to regular values, are an interesting general class of manifolds. Charts are provided by the implicit function theorem. Examples are the sphere in  $\mathbb{R}^n$  and the *special linear group*  $SL(n, \mathbb{R})$  of  $n \times n$  matrices with unit determinant. Viewing the space of  $n \times n$  matrices as  $\mathbb{R}^{n^2}$  we obtain  $SL(n, \mathbb{R})$  as the manifold defined by the equation  $\det A = 1$ . One can check that 1 is a regular value for the determinant. Thus this is a manifold defined by one equation. Examples of manifolds defined by several equations are the symplectic group of matrices  $A$  satisfying  $AJA^t = J$ , where  $J = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}$ , and the *orthogonal group* of matrices  $A$  satisfying  $AA^t = \text{Id}$ .

Let us quickly look how this notion behaves with respect to the basic constructions.

### 3. Submanifolds.

DEFINITION 4.5. A *submanifold*  $V$  of  $M$  (of dimension  $k \leq n$ ) is a differentiable manifold that is a subset of  $M$  such that the maximal atlas for  $M$  contains charts  $\{(U, h)\}$  for which the induced maps  $h|_{U \cap V}$  on  $U \cap V$  map to  $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$  and define charts for  $V$  compatible with the differentiable structure of  $V$ .

EXAMPLE 4.6. An open subset of a differentiable manifold  $M$  with the induced atlas is a submanifold of dimension  $n$ .

A function  $f: M \rightarrow \mathbb{R}$  is said to be *differentiable* if  $f \circ h^{-1}$  is differentiable on  $B \subset \mathbb{R}^n$  for any chart  $(U, h)$ . We denote by  $C^r(M)$  the functions that are  $C^r$  in this sense.

Manifolds defined by equations are by construction examples of submanifolds. Conversely every smooth manifold can be viewed as a submanifold of  $\mathbb{R}^n$  in a way described after Definition 4.8.

### 4. Direct products.

5. *Factor-spaces.* Identification spaces can also be smooth manifolds, for example, the unit circle viewed as  $\mathbb{R}/\mathbb{Z}$ , the torus  $\mathbb{R}^n/\mathbb{Z}^n$ , or compact factors of the hyperbolic plane. In either case the obvious charts that give these spaces the structure of a topological manifold are smoothly compatible. Conversely a smooth structure on a manifold always lifts to a smooth structure on any cover.

**b. Partition of unity and paracompactness.** An important result for analysis on manifolds is the fact that (using our assumption of second countability, that is, that there is a countable base for the topology) every smooth manifold admits a partition of unity, which is defined as follows and will be used later to define the volume element of a manifold:

DEFINITION 4.7. A *partition of unity* is a collection  $\{(U_i, \psi_i)\}_{i \in I}$ , where  $\{U_i\}$  is a locally finite open cover and  $\psi_i \in C^\infty(M)$  is nonnegative and has compact support in  $U_i$ , such that  $\sum_i \psi_i = 1$ . It is said to be *subordinate* to a cover  $\{O_j\}_{j \in J}$  if every  $U_i$  is contained in some  $O_j$ .

**c. Derivatives and the tangent bundle.** The derivative of functions can be calculated in coordinates, of course. But there is an invariant way of defining it using a local linear structure, that is, tangent vectors. This is obtained exactly by differentiating:

DEFINITION 4.8. Let  $M$  be a  $C^\infty$  manifold and  $p \in M$ . Consider curves  $c: (a, b) \rightarrow M$ , where  $a < 0 < b$ ,  $c(0) = p$ , such that  $h \circ c$  is differentiable at 0 for one (hence any) chart  $(U, h)$  with  $p \in U$ . Each such curve acts on  $C^\infty(M)$  by the *derivation* (that is, an operator satisfying the product rule)  $f \mapsto \frac{d}{dt}|_0 f \circ c$ . Many different curves will induce the same derivation and we identify two such curves. The space of these equivalence classes (at  $p$ ) obtained in this way, which is also the space of derivations at  $p$ , has a linear structure (since each derivation is a real-valued function) and turns out to have dimension  $n$ . It is called the *tangent space* at  $p$  of  $M$  and denoted  $T_p M$ . We denote the derivation, that is, tangent vector, induced by  $c: (a, b) \rightarrow M$  by  $\dot{c}(0)$  or  $\frac{d}{dt}|_0 c$ . Given a specific chart  $(U, h)$  we define the standard basis of  $T_p M$  by taking the canonical basis  $\{e_1, \dots, e_n\}$  of  $\mathbb{R}^n$  and letting  $\frac{\partial}{\partial x^i} := \dot{c}_i(0)$ , where  $c_i(t) = h^{-1}(h(p) + te_i)$ . We define the *tangent bundle* of

$M$  to be the disjoint union  $TM := \bigcup_{p \in m} T_p M$  of the tangent spaces with the *canonical projection*  $\pi: TM \rightarrow M$  such that  $\pi(T_p M) = \{p\}$ . Any chart  $(U, h)$  of  $M$  then induces a chart  $(U \times \bigcup_{p \in U} T_p U, H)$  by taking  $H(p, v) := (h(p), (v^1, \dots, v^n)) \in \mathbb{R}^n \times \mathbb{R}^n$ , where the  $v^i$  are the coefficients of  $v \in T_p M$  with respect to the basis  $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$  of  $T_p M$ . In this way  $TM$  is a differentiable manifold itself. A *vector field* is a map  $X: M \rightarrow TM$  such that  $\pi \circ X = \text{Id}_M$ , that is,  $X$  assigns to each  $p$  a tangent vector at  $p$ . We denote by  $\Gamma(M)$  the space of smooth vector fields on  $M$ . Thus vector fields act on functions as derivations. We write this as  $v f$  or  $v(f)$ . We shall see later that  $\mathcal{L}_v w := [v, w] := v w - w v$  is a derivation, that is, a vector field, and we call  $[v, w]$  the *Lie bracket* of  $v$  and  $w$  and  $\mathcal{L}_v$  the *Lie derivative*.

**d. Differentiable maps and diffeomorphisms.** We can now define the morphisms of the differentiable structure:

DEFINITION 4.9. Let  $M$  and  $N$  be differentiable manifolds. A map  $f: M \rightarrow N$  is said to be *differentiable* if for any charts  $(U, h)$  of  $M$  and  $(V, g)$  of  $N$  the map  $g \circ f \circ h^{-1}$  is differentiable on  $h(U \cap f^{-1}(V))$ . A differentiable map  $f$  acts on derivations by sending curves  $c: (a, b) \rightarrow M$  to  $f \circ c: (a, b) \rightarrow N$ . Differentiability means that curves inducing the same derivation have images inducing the same derivation. Thus we define the differential of  $f$  to be the map  $Df: TM \rightarrow TN$ ,  $\frac{d}{dt}|_0 c \mapsto \frac{d}{dt}|_0 f \circ c$  and the differential (or derivative)  $D_p f = Df|_p$  at  $p$  to be the restriction to  $T_p M$ . A *diffeomorphism* is a differentiable map with differentiable inverse. Two manifolds  $M, N$  are said to be *diffeomorphic* or *diffeomorphically equivalent* if and only if there is a diffeomorphism  $M \rightarrow N$ . An *embedding* of a manifold  $M$  in a manifold  $N$  is a diffeomorphism  $f: M \rightarrow V$  of  $M$  onto a submanifold  $V$  of  $N$ . We often abuse terminology and refer to an embedding of an open subset of  $M$  into  $M$  as a (*local*) *diffeomorphism* as well. An *immersion* of a manifold  $M$  into a manifold  $N$  is a differentiable map  $f: M \rightarrow V$  onto a subset of  $N$  whose differential is injective everywhere.

Clearly diffeomorphic manifolds are homeomorphic. The converse is, however, not true. There are “exotic” spheres and other manifolds whose differentiable structure is not diffeomorphic to the usual differentiable structure. Notice that an immersion need not be injective, for example, the unit circle in  $\mathbb{R}^2$  is an immersed line. But even injectively immersed manifolds may fail to be topological submanifolds. For example, nontrivial orbits of a flow are immersed lines, but the immersion topology of a nontrivially recurrent orbit (such as an orbit of a linear flow on  $\mathbb{T}^2$  with irrational slope) is not the same as the topology induced from the ambient space. Nevertheless we will refer to such objects as immersed submanifolds.

It turns out that any smooth manifold can be smoothly embedded into a Euclidean space, so every smooth manifold is, in fact, diffeomorphic to a submanifold of  $\mathbb{R}^n$ .

**e. Complex manifolds.** Complex manifolds are defined in a way very similar to the real ones by considering charts with values in  $\mathbb{C}^n$  instead of  $\mathbb{R}^n$  and requiring the coordinate changes between charts to be holomorphic. Since holomorphic maps are much more rigid than differentiable the resulting theory has many different aspects. For example the one-dimensional complex manifolds (Riemann surfaces) is a much richer subject than one- and even two-dimensional differentiable manifolds.

EXAMPLE 4.10. The *Riemann sphere*,  $\mathbb{C} \cup \{\infty\}$  which is homeomorphic to  $S^2$  becomes a one-dimensional complex manifold by considering an atlas of two charts  $(\mathbb{C}, \text{Id})$  and  $(\mathbb{C} \cup \{\infty\} \setminus \{0\}, I)$ , where  $I(z) = \begin{cases} \frac{1}{z} & \text{if } z \in \mathbb{C} \\ 0 & \text{if } z = \infty \end{cases}$ .

EXAMPLE 4.11. Identify  $\mathbb{R}^2$  with  $\mathbb{C}$  and consider the torus  $\mathbb{T}^2$  as the factor space  $\mathbb{C}/\mathbb{Z}^2$ .

## 5. Topology and geometry of surfaces

Compact (and some noncompact) surfaces are a favorite showcase for various branches of topology and geometry. They are two-dimensional topological manifolds with a variety of naturally defined differentiable structures, all diffeomorphic to each other. Complete topological classification is obtained via simple invariants which allow a variety of interpretations: combinatorial, analytical and geometrical.

Surfaces are also *one-dimensional* complex manifolds; but surprisingly the complex structures are not all equivalent (except for the case of the sphere), although they can be classified. This classification is the first result in a rather deep area at the junction of analysis, geometry and algebraic geometry known as *Teichmüller theory*.

In this section we review the classification of compact surfaces (2-dimensional manifolds) from various points of view. We start with a fundamental preparatory result which we will prove using tools from analysis.

### a. The Jordan Curve Theorem.

**DEFINITION 5.1.** A simple closed curve on a manifold  $M$  is a homeomorphic image of the circle  $S^1$  in  $M$ , or equivalently the image of  $S^1$  under a continuous injection  $S^1 \rightarrow M$ .

**LEMMA 5.2.** Any simple closed curve is nowhere dense in  $S^2$ .

**THEOREM 5.3. [Jordan Curve Theorem]** A simple closed curve  $C$  on the sphere  $S^2$  separates the sphere into two connected components, each homeomorphic to the disk, that is,  $S^2 \setminus C = \mathcal{D}_1 \cup \mathcal{D}_2$  with  $\mathcal{D}_1$  and  $\mathcal{D}_2$  homeomorphic to a disk.

Since the plane  $\mathbb{R}^2$  is homeomorphic to the sphere with one point removed then by Lemma 5.2 it is sufficient to prove the assertion of the Jordan Curve Theorem for the plane.

**b. Euler Theorem and genus.** Every orientable compact surface is homeomorphic to a space obtained from the sphere by attaching handles. Attaching a handle means deleting two disjoint disks and identifying the resulting two boundary circles with the boundary circles of a cylinder. The number  $g$  of attached handles is called the *genus* of the surface and is a topological invariant. As differentiable manifolds, surfaces are, in fact, determined up to diffeomorphism by their genus. The genus is related to the *Euler characteristic*  $\chi$  of a surface via  $\chi = 2 - 2g$ . The Euler characteristic can be described in various different ways. First consider a triangulation of the surface (see Definition 6.1), that is, a representation as a polyhedron with triangular faces, and let  $f$  be the number of faces,  $e$  the number of edges, and  $v$  the number of vertices. Then  $\chi = f - e + v$  independently of the triangulation. (In fact  $\chi = \beta_2 - \beta_1 + \beta_0$ , where the  $\beta_i$  are the Betti numbers of Definition 6.6. For the surface of genus  $g$  we have  $\beta_0 = \beta_2 = 1$  and  $\beta_1 = 2g$ , so we do get  $\chi = 2 - 2g$ .)

**c. Gauss–Bonnet theorem.** Second, we can consider a vector field  $v$  with finitely many fixed points. Then by the Poincaré–Hopf Index Formula (??) the sum of the indices of the fixed points of  $v$  is  $\chi$ . Finally let  $\langle \cdot, \cdot \rangle$  be a Riemannian metric on the surface and denote by  $K(x)$  the Gaussian curvature at the point  $x$ . Then we have

**THEOREM 5.4 (Gauss–Bonnet Theorem).**  $\chi = (1/2\pi) \int K(x) d \text{vol}$ , where  $\text{vol}$  is the volume induced by the Riemannian metric.

**d. The fundamental group.** The fundamental group of a surface can be represented in various ways. From the process of attaching handles one obtains generators  $a_i, b_i$  for  $i = 1, \dots, g$ , where each pair corresponds to a handle and  $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = \text{Id}$ . This representation also corresponds to the identifications made on the  $4g$ -gon to obtain the surface as an identification space. (For genus 1, the torus, this is the description as  $\mathbb{R}^2/\mathbb{Z}^2$ ; for genus 2, the double torus, this is similar to the description via the octagon given in Sections 5.4e and 14.4b, although the identifications are different.)

The universal cover of any orientable surface other than the sphere is  $\mathbb{R}^2$ .

More generally, the genus of a surface can be defined as the maximal number of disjoint closed curves such that the complement of their union is connected. This is easily visualized in terms of the description of genus by adding handles to a sphere.

**e. Intersection index.** For any two oriented closed curves on an orientable surface that have only transverse intersections we can define their intersection index by counting  $\pm 1$  according to whether their tangent vectors at an intersection point form a positively or negatively oriented pair. This index actually depends only on the homology classes of the curves and defines a nondegenerate skew-symmetric (symplectic) 2-form on  $H_1(M, \mathbb{R}) \times H_1(M, \mathbb{R})$ .

**f. Complex structure on surfaces.** We can also view surfaces as one-dimensional complex manifolds: The sphere is the Riemann sphere  $\mathbb{C} \cup \{\infty\}$ , the torus is the factor of  $\mathbb{C}$  by a lattice, and surfaces of higher genus are obtained from the upper half-plane  $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$  or the unit disk  $\mathbb{D}$  in  $\mathbb{C}$  as the factor of a group of Möbius transformations as described in ???. The Riemann sphere,  $\mathbb{R}^2$ , and the Poincaré disk each admit a metric of constant Gaussian curvature (positive, zero, and negative, correspondingly) and these descend to compact factors, so all compact surfaces admit a Riemannian metric of constant curvature.

**g. Nonorientable surfaces and surfaces with boundary.** Nonorientable surfaces are classified in a similar way. It is useful to begin with the best-known example, the *Möbius strip*, which is the nonorientable surface with boundary obtained by identifying two opposite sides of the unit square  $[0, 1] \times [0, 1]$  via  $(0, t) \sim (1, 1 - t)$ . Its boundary is a circle.

Any compact nonorientable surface is obtained from the sphere by attaching several *Möbius caps*, that is, deleting a disk and identifying the resulting boundary circle with the boundary of a Möbius strip. Attaching  $m$  Möbius caps yields a surface of genus  $2 - m$ . Alternatively one can replace any pair of Möbius caps by a handle, so long as at least one Möbius cap remains, that is, one may start from a sphere and attach one or two Möbius caps and then any number of handles.

All compact surfaces with boundary are obtained by deleting several disks from a closed surface. In general then a sphere with  $h$  handles,  $m$  Möbius strips, and  $d$  deleted disks has Euler characteristic  $\chi = 2 - 2h - m - d$ . In particular there is a finite list of surfaces with nonnegative Euler characteristic:

Surface	$h$	$m$	$d$	$\chi$	Orientable?
Sphere	0	0	0	2	yes
Projective plane	0	1	0	1	no
Disk	0	0	1	1	yes
Torus	1	0	0	0	yes
Klein bottle	0	2	0	0	no
Möbius strip	0	1	1	0	no
Cylinder	0	0	2	0	yes

## 6. Simplicial homology theory

### a. Simplicial complexes.

DEFINITION 6.1. For  $k, N \in \mathbb{N}$ ,  $v_0, \dots, v_k \in \mathbb{R}^N$  such that  $\{v_i - v_0\}_{i=1}^k$  is linearly independent, the convex hull  $\sigma$  of  $\{v_0, \dots, v_k\}$  is called the ( $k$ -dimensional) *simplex* spanned by  $\{v_0, \dots, v_k\}$  and the  $v_i$  are called the *vertices* of the simplex. The simplices spanned by a subset of  $\{v_0, \dots, v_k\}$  are called the *faces* of

Identifying two orderings of the set  $\{v_0, \dots, v_k\}$  of vertices of a simplex in  $M$  if they differ by an even permutation, we call a simplex with a choice of ordering of the vertices (modulo even permutations) an *oriented simplex*. The chosen orientation is then called the *positive orientation*, the other one the *negative orientation*. If  $\sigma$  is an oriented simplex then denote by  $-\sigma$  the same simplex with the negative orientation. Note that an orientation of a simplex induces an orientation on each face since the vertices of a face form a subset of the vertices of the simplex.

A (*finite*) *simplicial complex*  $S$  is a finite collection of simplices such that any two simplices intersect in a common face. The union of the simplices of  $S$  is denoted by  $|S|$ .

Any simplicial complex possess the natural topology of the factor space.

More generally, one considers infinite but locally finite simplicial complexes where any vertex belongs to finitely many simplices.

DEFINITION 6.2. A *triangulation* of a topological space  $M$  is a pair  $(S, h)$  consisting of a simplicial complex  $S$  and a homeomorphism  $h: S \rightarrow M$  from the simplicial complex  $S$  to  $M$ . The images of simplices, faces, and vertices under  $h$  are also called simplices, faces, and vertices.

A topological space together a triangulation is called a *simplicial polyhedron*

The following fact provides a crucial link between combinatorial and differential topology.

THEOREM 6.3. *Any compact differentiable manifold allows a triangulation.*

Now fix a triangulation  $(S, h)$  of  $M$  and consider formal sums  $\sum n_i \sigma_i$ , where  $\sigma_i$  are oriented  $k$ -simplices and  $n_i \in \mathbb{Z}$ . For  $n_i < 0$  we define  $n_i \sigma_i := (-n_i)(-\sigma_i)$ . The set of such formal sums with the obvious additive structure is the free group generated by the  $k$ -simplices and subject to the relations  $\sigma + (-\sigma) = 0$  and  $\sigma_i + \sigma_j = \sigma_j + \sigma_i$ , that is, a finitely generated free abelian group.

DEFINITION 6.4. A formal sum  $\sum n_i \sigma_i$  of oriented  $k$ -simplices is called a  *$k$ -chain*. Denote by  $C_k$  the set of  $k$ -chains. The *boundary operator*  $\partial: C_k \rightarrow C_{k-1}$  is defined by setting

$$\partial \sigma := \sum_{i=0}^k \sigma_i$$

for an oriented  $k$ -simplex  $\sigma$ , where the  $\sigma_i$  are the  $(k-1)$ -dimensional faces of  $\sigma$  with induced orientation, and extending linearly to  $C_k$ . For a triangulation  $S$  let

$$\chi := \sum_{k=0}^{\dim M} (-1)^k \text{card}\{\sigma \mid \sigma \text{ is a } k\text{-dimensional simplex in } S\}$$

It turns out that the number  $\chi$  is the same for different triangulations of a given manifold  $M$  and thus provides a topological invariant  $\chi(M)$  called the *Euler characteristic* of  $M$ .

An important combinatorial fact is that the boundary of a boundary is zero:

**THEOREM 6.5 (Poincaré Lemma).**  $\partial^2 = 0$ .

Since  $\partial: C_k \rightarrow C_{k-1}$  is by definition a homomorphism with respect to the additive structure, the Poincaré Lemma shows that the set  $B_k := \partial(C_{k+1}) \subset C_k$  of  $k$ -dimensional *boundaries* is a subgroup of the group  $Z_k := \ker \partial = \{c \in C_k \mid \partial c = 0\}$  of  $k$ -dimensional *cycles*. Since  $C_k$  is abelian,  $B_k$  is normal in  $C_k$ .

**DEFINITION 6.6.**  $H_k(M, \mathbb{Z}) := H_k := Z_k/B_k$  is called the *kth homology group* of  $M$  over the integers.  $H_k$ , as a finitely generated abelian group, can be written as  $\mathbb{Z}^{\beta_k} \oplus F$ , where  $F$  is a finite abelian group.  $\mathbb{Z}^{\beta_k}$  is then called the *free part* of  $H_k$ , and  $\beta_k$  is called the *kth Betti number*.

The zeroth homology group of a manifold always is  $\mathbb{Z}^n$ , where  $n$  is the number of connected components.

If we define the commutator subgroup  $[\pi_1(M, p), \pi_1(M, p)] := \{aba^{-1}b^{-1} \mid a, b \in \pi_1(M)\}$  then we have

**THEOREM 6.7 (Hurewicz).** *The first homology group  $H_1(M, \mathbb{Z})$  of a connected simplicial polyhedron  $M$  is isomorphic to the abelianization*

$$\pi_1(M, p)/[\pi_1(M, p), \pi_1(M, p)]$$

*of the fundamental group  $\pi_1(M, p)$  of  $M$  at any point  $p$ . The isomorphism is called the Hurewicz isomorphism.*

**PROPOSITION 6.8.**  $H_n(M, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } M \text{ is orientable,} \\ 0 & \text{if } M \text{ is not orientable,} \end{cases}$  where  $n = \dim M$ .

**PROOF.** Note that  $B_n = 0$  and thus  $H_n = Z_n$ . Suppose a simplex  $\sigma$  occurs  $i$  times in an  $n$ -cycle  $c$ . Since the boundary is zero, the  $(n-1)$ -faces of  $\sigma$  have to be canceled by other terms in  $\partial c$ . Thus each neighboring simplex has to appear  $i$  times with appropriate orientation. Thus  $c$  is the  $i$ -fold sum of all  $n$ -simplices of the triangulation, all oriented compatibly. In the nonorientable case the simplices cannot be ordered compatibly so that there is no  $n$ -cycle and  $H_n = Z_n = 0$ . In the orientable case there is exactly one such chain for each  $i \in \mathbb{Z}$ , so  $H_n = Z_n = \mathbb{Z}$ .  $\square$

**REMARK 6.9.** All homology groups are homotopy invariants.

**PROPOSITION 6.10 (Euler–Poincaré Formula).**  $\sum_{k=0}^{\dim M} (-1)^k \beta_k = \chi(M)$ , where  $\chi(M)$  is the Euler characteristic of  $M$ .

**EXAMPLE 6.11.** For  $S^m$  the Betti numbers are  $\beta_0 = \beta_m = 1$  and  $\beta_k = 0$  for  $0 < k < m$ .

**EXAMPLE 6.12.** Since the fundamental group  $\mathbb{Z}^2$  of  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  is abelian, it coincides with the first homology group, that is,  $\beta_1 = 2$ . The other Betti numbers are  $\beta_0 = 1$  (number of connected components) and  $\beta_2 = 1$  (since  $\mathbb{T}^2$  is orientable). For  $\mathbb{T}^n$  one has  $\beta_k = \binom{n}{k}$ .

EXAMPLE 6.13. For the ball the only nonzero Betti number is  $\beta_0 = 1$  since it is homotopic to a point.

In order to understand the behavior of a map  $f$  with respect to the homology we need to adapt  $f$  to a given triangulation.

Suppose  $S$  and  $T$  are two simplicial complexes triangulating  $M$ . A map  $s: T \rightarrow S$  that maps simplices of  $T$  linearly onto simplices of  $S$  is then called a *simplicial map*.  $T$  is called a *refinement* of  $S$  if every simplex of  $S$  is triangulated by simplices of  $T$ . As subsets of  $\mathbb{R}^n$ ,  $S$  and  $T$  are then naturally identified and we will assume that they triangulate  $M$  via a common homeomorphism  $h$ .

THEOREM 6.14. *For any map  $f: M \rightarrow M$  and any triangulation  $S$  of  $M$  there exists a refinement  $S'$  of  $S$  and a simplicial map  $s: S' \rightarrow S$  homotopic to  $h^{-1} \circ f \circ h$ .*

Such a map  $S$  is called a *simplicial approximation* of  $f$ . Via it  $f$  acts on homology: Note that a  $k$ -chain  $c_k = \sum a_i \sigma_i$  on  $S$  induces a  $k$ -chain  $c'_k$  on  $S'$  by replacing every simplex  $\sigma_i$  by a  $k$ -chain in  $S'$  triangulating it. The simplicial map  $s$  sends  $c'_k$  to a  $k$ -chain  $s_* c'_k$  on  $S$  again. It can be shown that this induces an action on the  $k$ th homology which is independent of the choice of simplicial approximation. Thus, in particular in the case of an orientable manifold  $M$  when  $H_n(M, \mathbb{Z}) = \mathbb{Z}$ , a map  $f$  induces a homomorphism  $f_*: \mathbb{Z} \rightarrow \mathbb{Z}$ .

More generally the construction of the homology groups can be carried out over any commutative ring  $R$  instead of  $\mathbb{Z}$  to give homology groups  $H_k(M, R)$  which are modules over  $R$ . One starts with formal sums of simplices with coefficients from  $R$  and proceeds as before. In particular, if  $\mathbb{K}$  is a field then  $H_k(M, \mathbb{K})$  is isomorphic to  $\mathbb{K}^{\beta_k}$  and is thus determined by the free part of the homology group over  $\mathbb{Z}$ . As will be seen in Section 7 the  $k$ th de Rham cohomology group is naturally isomorphic via integration over cycles to the dual of  $H_k(M, \mathbb{R})$ .

## 7. Differentiable manifolds

We return to the general discussion of differentiable manifolds started in Section 4

**a. Vector fields, flows and differential operators.** By the theorems of existence, uniqueness, and smooth dependence for solutions of ordinary differential equations a  $C^1$  vector field on  $M$  induces a local flow, that is, for every  $p \in M$  there is a curve  $c_{v,p}: (-\epsilon, \epsilon) \rightarrow M$  such that  $c_{v,p}(0) = p$  and  $\dot{c}_{v,p}(t) := \frac{d}{dt}c_{v,p}(t) = v(c_{v,p}(t))$ . Here  $\epsilon$  can be chosen to depend continuously on  $p$ . Where defined the map  $\varphi_v: (p, t) \mapsto \varphi^t(p) := c_{v,p}(t)$  is as smooth as  $v$ . By continuity of  $\epsilon$  it is bounded on any compact manifold and hence by the group property  $c_{v,p}(t+s) = c_{v(c_{v,p}(t)), c_{v,p}(t)}(s)$  (which follows from uniqueness) every vector field on a compact manifold induces a *complete* flow, that is,  $\varphi_v^t$  is defined for *all* times. If  $\varphi_v^t$  and  $\varphi_w^s$  are the flows for vector fields  $v$  and  $w$ , respectively, then usually the diffeomorphisms  $\varphi_v^t$  and  $\varphi_w^s$  do not commute, that is,  $\varphi_v^t \circ \varphi_w^s \neq \varphi_w^s \circ \varphi_v^t$ . If they do, the vector fields  $v$  and  $w$  are said to *commute*. The extent to which two vector fields  $v, w$  fail to commute is measured by their Lie bracket  $[v, w]$  which can be computed as  $[v, w](p) = \lim_{t \rightarrow 0} (w - d\varphi_v^t w)(\varphi_v^t(p))/t$ .

Let us now show briefly how these invariant notions appear in local coordinates. If  $(U, h)$  is a chart then we say that we have coordinates  $(x^1, \dots, x^n)$  on  $U$ . For  $p \in U$  the canonical basis of  $T_p M$  is the set of *derivations*  $\partial/\partial x^i$  induced by the curves  $c_i(t) := h^{-1}(h(p) + te_i)$ , where  $e_i$  is the  $i$ th standard basis vector in  $\mathbb{R}^n$ . A tangent vector  $v \in T_p M$  can then be written as  $v = \sum_{i=1}^n v^i \partial/\partial x^i$  and if  $f: M \rightarrow \mathbb{R}$  is smooth then  $vf = \sum_{i=1}^n v^i \partial(f \circ h^{-1})/\partial x^i$ . Thus the induced coordinates of  $TM$  are  $(x^1, \dots, x^n, v^1, \dots, v^n)$ , where the  $v^i$  are the components we just defined. Likewise a vector field is locally given by a representation  $v(p) = \sum_{i=1}^n v^i(p) \partial/\partial x^i$  and it is smooth if and only if the  $v^i$  are. To see that the Lie bracket of two vector fields  $v, w$  defines a derivation, that is, a vector field, we calculate in local coordinates. Namely, write  $v = \sum_{i=1}^n v^i \partial/\partial x^i$ ,  $w = \sum_{i=1}^n w^i \partial/\partial x^i$  and for convenience write  $f$  for  $f \circ h$ . Then using the theorem of H. A. Schwarz that second partial derivatives commute we obtain

$$\begin{aligned} (vw - wv)f &= v \sum_{i=1}^n w^i \frac{\partial f}{\partial x^i} - w \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i} \\ &= \sum_{i,j=1}^n v^j \frac{\partial w^i}{\partial x^j} \frac{\partial f}{\partial x^i} + \sum_{i,j=1}^n v^j w^i \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{i,j=1}^n w^j \frac{\partial v^i}{\partial x^j} \frac{\partial f}{\partial x^i} + \sum_{i,j=1}^n v^i w^j \frac{\partial^2 f}{\partial x^j \partial x^i} \\ &= \sum_{i,j=1}^n \left( v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \right) \frac{\partial f}{\partial x^i}, \end{aligned}$$

that is,  $[v, w]$  is indeed a vector field given locally by  $v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}$ . In particular  $[\partial/\partial x^i, \partial/\partial x^j] = 0$ . There are several important properties of Lie brackets that are not hard to check in local coordinates. By definition we obviously have  $[v, w] = -[w, v]$  and  $[\cdot, \cdot]$  is  $\mathbb{R}$ -bilinear, that is,  $[\alpha v + \beta w, z] = \alpha[v, z] + \beta[w, z]$  for  $\alpha, \beta \in \mathbb{R}$ . Next observe that for functions as coefficients we get  $[fv, gw] = fg[v, w] + f(vg)w - g(wf)v$  by a coordinate calculation similar to the preceding one. This means in particular (for  $f \equiv 1$ ) that the Lie derivative is a derivation, that is, satisfies the product rule  $\mathcal{L}_v(gw) =$

$g\mathcal{L}_v w + \mathcal{L}_v g w$ . Furthermore there is the fundamental *Jacobi identity*

$$(7.1) \quad [v, [w, z]] + [w, [z, v]] + [z, [v, w]] = 0.$$

This is straightforward in coordinates. Namely, we know that only first-order derivatives occur, so we may simplify the calculation by discarding all higher-order derivatives. The symmetry then makes the remaining terms cancel. Alternatively write  $[v, w] = vw - wv$  and expand (7.1) accordingly to see that all terms cancel.

Differentiating differentiable maps between manifolds is also straightforward calculus on local coordinates: If  $f: M \rightarrow N$  and  $(U, h), (V, k)$  are local charts around  $p \in M$  and  $f(p) \in N$ , respectively, then the differential of  $f$  at  $p$  is represented by the matrix of partial derivatives of the map  $k \circ f \circ h^{-1}$  in Euclidean space with respect to the standard bases.

**b. Tensor bundles.** The tangent bundle is an example of the following:

**DEFINITION 7.1.** A differentiable *vector bundle* with *structure group*  $G$ , a subgroup of  $GL(m, \mathbb{R})$ , over  $M$  (the *base space*) is a manifold  $P$ , called the *total space* or *bundle space*, such that the projection  $\pi: P \rightarrow M$  is differentiable and furthermore locally  $P = M \times \mathbb{R}^m$ , that is, every  $x \in M$  has a neighborhood  $U$  such that there is a diffeomorphism  $h: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^m$ ,  $u \mapsto (\pi(u), \varphi(u))$  and such that for any point  $x$  in the intersection  $U_1 \cap U_2$  of two neighborhoods the trivialization differs by an element of  $G$ . A *subbundle* or *distribution* is a bundle whose fibers are contained in those of  $P$ . For two distributions  $E, F$  we define the *Whitney sum*  $E + F$  to be the distribution with  $(E + F)_p = E_p + F_p$ . We use “ $\oplus$ ” if the sum is (pointwise) direct, that is,  $E_p \cap F_p = \{0\}$  for all  $p \in M$ . A *section* of  $P$  is a map  $v: M \rightarrow P$  such that  $\pi \circ v = \text{Id}_M$ .

**EXAMPLE 7.2.** The tangent bundle  $TM$  of  $M$  is of this form: Here  $m$  is the dimension of  $M$  and  $G = GL(m, \mathbb{R})$  acts by the linear coordinate changes in the tangent fibers induced by coordinate change in the base. The sections are the vector fields. If there is a nonvanishing vector field on  $M$  then the one-dimensional subspaces it spans at every point define a one-dimensional distribution.

Note that the differentiable manifold  $TM$  has in turn a tangent bundle  $TTM$ . This is an important object. On one hand it allows us to differentiate vector fields. On the other hand classical mechanics involves second-order differential equations and the natural setting for second derivatives is the second (or double) tangent bundle  $TTM$ .

The second tangent bundle  $TTM$  is obviously a vector bundle over  $TM$ , but it is, in fact, a vector bundle over  $M$  as well. To that end notice that coordinate changes in  $M$  change coordinates in  $TTM$  by a coordinate change determined again by the linear part of the coordinate change in  $M$ . We will return to this in the setting of Riemannian manifolds.

From the linear structure in the tangent spaces arise linear objects other than vectors and linear maps (for example, differentials). Namely, it is often important to consider *multilinear* maps. The easiest examples, and a building block, are *1-forms*.

**DEFINITION 7.3.** We denote by  $T^*M$  the *cotangent bundle* consisting of the spaces  $T_p^*M = (T_p M)^*$  of linear maps (covectors)  $T_p M \rightarrow \mathbb{R}$ . A section of  $T_p^*M$  is called a *1-form*. A multilinear map  $\underbrace{T_p^*M \oplus \cdots \oplus T_p^*M}_{k \text{ times}} \oplus \underbrace{T_p M \oplus \cdots \oplus T_p M}_{l \text{ times}} \rightarrow \mathbb{R}$  (that is,

linear in each entry independently) is called a  $(k, l)$ -*tensor*. A section of the bundle  $TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M = (TM)^{\otimes k} \otimes (T^*M)^{\otimes l}$  is a  $(k, l)$ -*tensor field* (or

tensor). A tensor is called *smooth* if its values on smooth vector and covector fields define a smooth function. (Alternatively, if its coefficients in local coordinates are smooth.)

Thus a vector is a  $(1, 0)$ -tensor, a 1-form is a  $(0, 1)$ -tensor, and the Riemannian metrics defined in ?? are  $(0, 2)$ -tensors. A basis for the space of 1-forms on  $T_p M$  is given by the forms  $dx^i$  which are given by the derivatives of the coordinate functions  $x^i$ , that is,

$$dx^i(\partial/\partial x^j) = \delta_j^i := \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \text{ The derivative of a function } f \text{ is a 1-form } Df(v) :=$$

$vf = \sum_{i=1}^n \partial f/\partial x_i dx^i$ . If  $T$  is a  $(k, l)$ -tensor then  $T = T_{i_1, \dots, i_l}^{j_1, \dots, j_k} \partial/\partial x^{j_1} \otimes \dots \otimes \partial/\partial x^{j_k} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_l}$  with  $T_{i_1, \dots, i_l}^{j_1, \dots, j_k} = T(dx^{j_1}, \dots, dx^{j_k}, \partial/\partial x^{i_1}, \dots, \partial/\partial x^{i_l})$ . There is a natural way to extend the Lie derivative to tensors. Namely, note first that for  $(1, 0)$ -tensors (vector fields) it is already defined and that for  $(0, 0)$ -tensors (functions) we can define  $\mathcal{L}_v f := vf$ . Now extend to  $(0, 1)$ -tensors  $\xi$  by setting  $\mathcal{L}_v(\xi(w)) = \mathcal{L}_v(\xi)(w) + \xi(\mathcal{L}_v w)$ . Likewise one can extend  $\mathcal{L}_v$  to any tensor field by postulating the product rule  $\mathcal{L}_v(\xi \otimes \eta) = \mathcal{L}_v \xi \otimes \eta + \mathcal{L}_v \eta \otimes \xi$ . If  $\omega$  is a  $(0, 1)$ -tensor on  $N$  and  $f: M \rightarrow N$  differentiable then we can define the *pullback*  $f^* \omega$  of  $\omega$  on  $M$  by  $f^* \omega(v) := \omega(Dfv)$ . This, of course, works for  $(0, k)$ -tensors just as well. Likewise one can send vectors from  $M$  to  $N$  via  $Df$ , but this can be expected to send vector fields to vector fields only if  $f$  is injective (if  $f(p) = f(q)$  and  $v$  is a vector field such that  $Dfv(p) \neq Dfv(q)$  then there is no well-defined vector field " $f_* v$ " on  $f(M)$ ). If  $f$  is a diffeomorphism then this is no problem, however. Using pullbacks the Lie derivative of a  $(0, k)$ -tensor can be computed by using the flow  $\varphi^t$  defined by the vector field  $v$  to write

$$\mathcal{L}_v \omega = \lim_{t \rightarrow 0} (1/t)((\varphi^t)^* \omega - \omega).$$

The Lie derivative of any  $(k, l)$ -tensor can be computed similarly.

An important special class of tensors is that of alternating ones:

**DEFINITION 7.4.** A  $(0, k)$ -tensor  $\omega$  on a linear space is said to be an *alternating tensor* or an (*exterior*) *form* if  $\omega(v_1, \dots, v_k) = 0$  whenever  $v_i = v_j$  for some  $i \neq j$ . A  $(0, k)$ -tensor field is said to be *alternating* if it is alternating at every point. Alternating  $(0, k)$ -tensor fields are called *k-forms*, and the space of *k-forms* is denoted by  $\Gamma(\bigwedge^k T^* M)$ . In analogy to the asymmetric part of a matrix the alternating part  $\mathcal{A}\eta$  of a  $(0, k)$ -tensor  $\eta$  is defined by  $\mathcal{A}\eta = 1/k! \sum_{\pi \in S_k} \text{sgn } \pi \eta \circ \pi$ , where  $\pi$  permutes the entries and  $\text{sgn } \pi$  is its sign, that is,  $-1$  if  $\pi$  is odd,  $1$  otherwise. Thus  $\mathcal{A}$  is a projection of  $(T^* M)^{\otimes k}$  to  $\bigwedge^k T^* M$ . We define the *wedge product* or *exterior product* of  $\omega \in \bigwedge^k T^* M$  and  $\eta \in \bigwedge^l T^* M$  by

$$\omega \wedge \eta := \frac{(k+l)!}{k! l!} \mathcal{A}(\omega \otimes \eta) \in \bigwedge^{k+l} T^* M.$$

Nonzero elements of  $\Gamma(\bigwedge^n T^* M)$  are called *volume elements* and two volume elements  $\Omega, \Omega'$  are said to be *equivalent* if  $\Omega' = f\Omega$  for some  $f \in C^\infty(M)$ ,  $f > 0$ . An equivalence class of volume forms is called an *orientation* of  $M$  and  $M$  is called *orientable* if there exists an orientation on  $M$ .

With these definitions one gets the following standard facts:  $\omega \wedge \eta$  is  $\mathbb{R}$ -bilinear in  $\omega$  and  $\eta$ ,  $\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta$  (hence  $\omega \wedge \omega = 0$  for odd  $k$ ),  $f^*(\omega \wedge \eta) = (f^* \omega) \wedge (f^* \eta)$ , and  $\omega \wedge (\eta \wedge \lambda) = (\omega \wedge \eta) \wedge \lambda =: \omega \wedge \eta \wedge \lambda$ . A basis for  $\bigwedge^k T_p^* M$  is given by  $\{dx^{i_1} \wedge \dots \wedge dx^{i_k} \mid 1 \leq i_j \leq n\}$ , where  $\{dx^i \mid 1 \leq i \leq n\}$  is the dual basis for  $\{\partial/\partial x^i \mid 1 \leq i \leq n\}$ . Thus

$\dim \bigwedge^k T_p^* M = \binom{n}{k}$ . In fact,  $\beta^1 \wedge \cdots \wedge \beta^k \neq 0$  if and only if  $\{\beta^1, \dots, \beta^k\} \subset T_p^* M$  is linearly independent.

A manifold is orientable if and only if  $\Gamma(\bigwedge^n T^* M)$  is one-dimensional over  $C^\infty(M)$ . (Namely, there exists a volume, hence the dimension is at least one, and for two volumes  $\Omega$  and  $\Omega'$  the function  $\varphi := \Omega'/\Omega$  is well defined, since  $\Gamma(\bigwedge^n T_p^* M)$  is one-dimensional, and smooth as well.) One can also check that orientability is equivalent to the existence of an oriented atlas, that is, an atlas where  $h \circ h'$  preserves the orientation of  $\mathbb{R}^n$  for any two charts  $h, h'$ . On a compact manifold a volume form can be integrated to give the total volume. This is done via charts as follows. In  $\mathbb{R}^n$  we define  $\int \Omega := \int \Omega_{1, \dots, n} dx^1 \cdots dx^n$  for any volume  $\Omega = \Omega_{1, \dots, n} dx^1 \wedge \cdots \wedge dx^n$ . For orientation-preserving diffeomorphisms  $f$  we get  $\int f^* \Omega = \int \Omega$ . Thus we can define  $\int \Omega$  for a manifold  $M$  by taking a partition of unity  $\{U_i, \psi_i\}$  subordinate to a covering by charts  $(V_i, h_i)$  and define  $\int \Omega := \sum_i \int (h_i)_*(\psi_i \Omega)$ , and this definition via charts is coordinate independent.

**c. Exterior calculus.** Next we want to study the calculus of exterior forms, also called exterior calculus.

**DEFINITION 7.5.** The *exterior derivative*  $d: \Gamma(\bigwedge^k T^* M) \rightarrow \Gamma(\bigwedge^{k+1} T^* M)$  (for any  $k$ ) is defined by the following axioms (which uniquely determine  $d$ ):  $df = Df$  for functions,  $d$  is  $\mathbb{R}$ -linear and  $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ ,  $d \circ d = 0$ , and  $d$  is locally defined, that is, if two forms coincide on an open set  $O$  then their derivatives coincide on  $O$  as well.

By induction on dimension one sees that this is well defined. Namely, if  $\omega = \varphi d\psi^1 \wedge \cdots \wedge d\psi^k$  then necessarily  $d\omega = d\varphi \wedge d\psi^1 \wedge \cdots \wedge d\psi^k$ . The last property is also satisfied inductively since it holds for functions:  $dd\varphi = \sum_{i,j=1}^n (DD\varphi)_{ij} dx^i \wedge dx^j = \frac{\partial^2 \varphi}{\partial x^i \partial x^j} dx^i \wedge dx^j = 0$ . Furthermore  $d$  commutes with pullback and the Lie derivative:  $f^* d\omega = d(f^* \omega)$  (and  $f_* d\omega = d(f_* \omega)$  if  $f$  is a diffeomorphism) and  $\mathcal{L}_v(\omega^1 \wedge \cdots \wedge \omega^k) = \mathcal{L}_v \omega^1 \wedge \cdots \wedge \omega^k + \omega^1 \wedge \cdots \wedge \mathcal{L}_v \omega^k$ , whence  $d\mathcal{L}_v = \mathcal{L}_v d$ .

We occasionally use the convenient notation of the *contraction* of  $\omega$  with a vector  $v$  defined by  $v \lrcorner \omega := \omega(v, \cdot, \dots, \cdot)$ . This is  $\mathbb{R}$ -linear and  $C^\infty(M)$ -linear in  $v$ . Furthermore  $v \lrcorner (\omega \wedge \eta) = (v \lrcorner \omega) \wedge \eta + (-1)^k \omega \wedge (v \lrcorner \eta)$  and  $v \lrcorner df = \mathcal{L}_v f$  and

$$(7.2) \quad \mathcal{L}_v \omega = v \lrcorner d\omega + d(v \lrcorner \omega).$$

Finally  $f^* v \lrcorner f^* \omega = f^*(v \lrcorner \omega)$  and  $f_* v \lrcorner f_* \omega = f_*(v \lrcorner \omega)$  for any diffeomorphism  $f$ .

Associated with forms is a cohomology theory which is based on the following notion and theorem:

**DEFINITION 7.6.**  $\omega \in \Gamma(\bigwedge^k T^* M)$  is said to be *closed* if  $d\omega = 0$  and *exact* if  $\omega = d\eta$  for some  $\eta \in \Gamma(\bigwedge^{k-1} T^* M)$ .

Since  $d^2 = 0$  every exact form is closed. Locally the converse holds:

**THEOREM 7.7 (Poincaré Lemma).** *If  $\omega$  is closed then for all  $p \in M$  there is a neighborhood  $U$  of  $p$  on which  $\omega$  is exact.*

**PROOF.** We use the homotopy trick (see ??, ??): Assume  $p = 0 \in \mathbb{R}^n$  and let  $v_t(x) = x/t$ .  $v_t$  generates the flow  $\varphi^t(x) = tx$  for  $t > 0$ , so  $d/dt(\varphi^t)^* \omega = (\varphi^t)^* \mathcal{L}_{v_t} \omega = (\varphi^t)^*(d(v_t \lrcorner \omega)) = d((\varphi^t)^*(v_t \lrcorner \omega))$  since  $d\omega = 0$ , and  $\omega - (\varphi^{t_0})^* \omega = d \int_{t_0}^1 (\varphi^t)^*(v_t \lrcorner \omega) dt \xrightarrow[t_0 \rightarrow 0]{} 0$

$d \int_0^1 (\varphi^t)^* v_t \lrcorner \omega dt =: d\eta$ . The limit exists since  $(\varphi^t)^*(v_t \lrcorner \omega)_x(w_x) = \omega_{tx}(x/t, tw_x) = \omega_{tx}(x, w_x)$ .  $\square$

The advertised cohomology theory for compact manifolds is the *de Rham cohomology*: The  $k$ th cohomology group is the factor of the space of closed  $k$ -forms by the space of exact  $k$ -forms. This is a finite-dimensional vector space by virtue of the Poincaré Lemma. It is naturally dual to the  $k$ th homology group with real coefficients (see Section 6 of the manifold. It also possesses a natural multiplicative structure induced by the wedge product.

To obtain this duality we need to define integrals of forms. To that end notice first that the integral of a  $k$ -form over an immersed  $k$ -simplex can be defined as the integral of the pullback by the immersion. By the change of variables formula this depends only on the image of the immersion. A useful result for integration of forms is the theorem of Stokes:

**THEOREM 7.8.** *If  $M$  is an  $n$ -manifold, possibly with boundary, and  $\omega$  an  $(n-1)$ -form on  $M$  then  $\int_M d\omega = \int_{\partial M} \omega$ .*

In particular an exact form on any boundaryless manifold integrates to zero. Now any immersed  $k$ -dimensional submanifold can be partitioned (up to overlapping boundaries) into immersed  $k$ -simplices (triangulation), so the integral of a  $k$ -form over immersed  $k$ -dimensional submanifolds is well defined (again, this is independent of the triangulation). In particular  $k$ -forms can be integrated over embedded  $k$ -cycles (see Section 6). It is convenient to think of cycles as embedded boundaryless manifolds, and indeed by the Stokes Theorem these integrals depend only on the cohomology class of the form. On the other hand for a given form these integrals depend only on the homology class (with real coefficients) of the cycle because two homologous manifolds form the boundary of a manifold. This gives the duality between the de Rham cohomology of forms and the simplicial homology of the manifold.

We shall have occasion to invoke a result related to the preceding ones:

**LEMMA 7.9.** *On an  $n$ -manifold an  $n$ -form with zero integral is exact.*

**d. Transversality.** On a smooth manifold we have a natural notion of “measure zero” because if the image of a set in one chart has measure zero then the same holds for any chart. If  $f: M \rightarrow N$  is a differentiable map then  $x \in N$  is called a *regular value* if for all  $y \in f^{-1}(\{x\})$  the differential  $Df_y$  has maximal rank, that is, its rank is  $\min(\dim M, \dim N)$ . Otherwise  $x$  is called a *singular value*.

**THEOREM 7.10 (Sard Theorem).** *If  $f \in C^\infty(M, N)$  then the set of singular values of  $f$  has Lebesgue measure zero.*

One implication is that by the Implicit Function Theorem almost any value of a smooth  $f$  has a manifold as a level set. This theorem is local because we can take  $M$  to be a neighborhood of a point in some manifold. Sard’s Theorem is the central background result for transversality theory.

**DEFINITION 7.11.** Let  $M$  be a smooth manifold and  $K, N \subset M$  smooth submanifolds.  $K$  and  $N$  are said to be *transverse* at  $x \in M$  if  $x \notin K \cap N$  or  $T_x K + T_x N = T_x M$ . We write  $K \pitchfork_x N$ . In particular, if  $\dim K + \dim N = \dim M$  and  $x \in K \cap N$  the latter condition is equivalent to  $T_x K \cap T_x N = \{0\}$ .

We say that  $K$  and  $N$  are transverse (to each other), written  $K \pitchfork N$ , if  $K \pitchfork_x N$  for all  $x \in K \cap N$ . If  $K$  and  $M$  are manifolds with boundary (see Definition 1.44) then they

are said to be transverse if the (boundaryless) manifolds  $\partial K$ ,  $K \setminus \partial K$ ,  $\partial M$ ,  $M \setminus \partial M$  are pairwise transverse in the previous sense.

Transversality is a  $C^1$ -open condition:

PROPOSITION 7.12. *Transverse intersections are stable in the  $C^1$  topology.*

Let us first observe that transverse intersections can be brought into a “normal form”:

LEMMA 7.13 (Adapted coordinates). *If  $K \pitchfork_x N$  in  $M$ ,  $x \in K \cap N$ , and  $k = \dim K$ ,  $n = \dim N$ ,  $m = \dim M$  then there is a neighborhood  $U$  of  $x$  and coordinates  $(x_1, \dots, x_m)$  on  $U$  such that in these coordinates*

$$K \cap U = \{(x_1, \dots, x_m) \mid x_{k+1} = \dots = x_m = 0\}$$

and

$$N \cap U = \{(x_1, \dots, x_m) \mid x_1 = \dots = x_{m-n} = 0\}.$$

PROOF. By extending a basis for  $T_x K \cap T_x N$  to a basis for  $T_x K$  and a basis for  $T_x N$  one obtains local coordinates  $\psi: U \rightarrow \mathbb{R}^m$  on a neighborhood  $U$  of  $x$  in which  $D_x \psi T_x K = \{v \in \mathbb{R}^m \mid v_{k+1} = \dots = v_m = 0\}$  and  $D_x \psi T_x N = \{v \in \mathbb{R}^m \mid v_1 = \dots = v_{m-n} = 0\}$ . Thus, after possibly shrinking  $U$ , there exists  $\varphi: \mathbb{R}^k \rightarrow \mathbb{R}^{m-k}$  such that  $\psi(K \cap U) = \text{graph } \varphi$ . Then  $\phi: (x_1, \dots, x_m) \mapsto (x_1, \dots, x_m) - (0, \dots, 0, \varphi(x_1, \dots, x_k))$  has the effect that  $\phi \circ \psi$  represents  $K$  as required. “Straightening out”  $N$  similarly yields the claim.  $\square$

PROOF. of the proposition We will use the Implicit Function Theorem. Since the problem is a local one we pass to adapted coordinates and note that  $C^1$ -perturbations  $K'$ ,  $N'$  of  $K$  and  $N$  are graphs of maps  $\xi: \mathbb{R}^k \times \{0\} \rightarrow \mathbb{R}^{m-k}$  and  $\eta: \{0\} \times \mathbb{R}^n \rightarrow \mathbb{R}^{m-n}$ , respectively. The space of pairs of  $C^1$  maps of this kind is a Banach space  $E$ . So is the space  $F = \mathbb{R}^{m-n} \times \{0\} \times \mathbb{R}^{m-k}$ . Consider the map

$$f: E \times F \rightarrow F, \quad ((\xi, \eta), (x, 0, y)) \mapsto (x, 0, \xi(x, 0, 0)) - (\eta(0, 0, y), 0, y).$$

It vanishes at 0, and the derivative at 0 in the  $F$ -direction is nonsingular by transversality. The Implicit Function Theorem yields  $x_{(\xi, \eta)}$  and  $y_{(\xi, \eta)}$  such that  $f((\xi, \eta), (x_{(\xi, \eta)}, 0, y_{(\xi, \eta)})) = 0$  or  $(x_{(\xi, \eta)}, 0, \xi(x_{(\xi, \eta)}, 0, 0)) = (\eta(0, 0, y_{(\xi, \eta)}), 0, y_{(\xi, \eta)})$ . Therefore  $K'$  and  $N'$  do intersect. Transversality of the intersection follows because in local coordinates the tangent spaces to  $K'$  and  $N'$  are small perturbations of those of  $K$  and  $N$  and the spanning condition is open.  $\square$

Here is an immediate consequence of Lemma 7.13:

COROLLARY 7.14. *If  $K$  and  $M$  are compact transverse manifolds (possibly with boundary) then any sufficiently small  $C^1$ -perturbations  $\tilde{K}$  and  $\tilde{M}$  are transverse.*

DEFINITION 7.15. Let  $0 \leq r \leq \infty$  and  $M$  a  $C^r$  manifold. Two submanifolds  $K_1$  and  $K_2$  of  $M$  are said to be  $C^r$ -close if there exist a  $C^r$  manifold  $K_0$  and  $C^r$  embeddings  $f_i: K_0 \rightarrow K_i$  such that  $f_1$  and  $f_2$  are  $C^r$ -close.

THEOREM 7.16 (Transversality Theorem). *Let  $M$  be a  $C^\infty$  manifold of dimension  $m$ , and  $N \subset M$  a submanifold of dimension  $n$ . Then among the  $k$ -dimensional submanifolds  $K \subset M$ , those transverse to  $N$  are  $C^\infty$ -dense.*

PROOF. Consider a coordinate neighborhood  $U$  such that

$$N \cap U = \{(x_1, \dots, x_m) \mid x_1 = \dots = x_{m-n} = 0\}$$

and let  $T$  be the natural transversal

$$T := \{(x_1, \dots, x_m) \mid x_{m-n+1} = \dots = x_m = 0\}.$$

For  $s = (s_1, \dots, s_{m-n})$  let furthermore

$$N^s := \{(x_1, \dots, x_m) \mid x_1 = s_1, \dots, x_{m-n} = s_{m-n}\}$$

and let  $\pi$  be the projection of  $U$  onto  $T$  along  $N^0 = N \cap U$ , that is,

$$\pi(x_1, \dots, x_m) = (x_1, \dots, x_{m-n}, 0, \dots, 0).$$

Now let  $K$  be represented as the image of the  $C^\infty$  embedding  $\varphi: K_0 \rightarrow M$  and let  $\tilde{K}_0 = \varphi^{-1}(K \cap U)$ . Consider the  $C^\infty$  map  $\pi \circ \varphi: \tilde{K}_0 \rightarrow T$ . Whenever the point  $\{(s_1, \dots, s_{m-n}, 0, \dots, 0)\} = T \cap N^s$  is a regular value of  $\pi \circ \varphi$  then  $(K \cap U) \pitchfork N^s$ . By the Sard Theorem the set of regular values has full Lebesgue measure in  $T$ , whence there are regular values arbitrarily close to 0. Thus arbitrarily close to  $N \cap U$  there are manifolds transverse to  $K$ .

To extract a global result from these considerations we first take a slightly smaller neighborhood  $U' \subset U$  and pick a nonnegative  $C^\infty$  bump function  $\rho$  that is 1 on  $U'$  and vanishes outside  $U$ . Then for any  $s \in \mathbb{R}^{m-n}$  one constructs a  $C^\infty$  vector field given by  $\rho \sum_{i=1}^{m-n} s_i \frac{\partial}{\partial x_i}$  in  $U$  and 0 outside  $U$ . Let  $T_s$  be the time-one map for the flow generated by this vector field. For any  $r \in \mathbb{N}$  the diffeomorphism  $T_s$  is  $C^r$ -close to the identity as long as  $\|s\|$  is sufficiently small. For such  $s$

$$\tilde{N}^s := (T_s \circ \varphi)(N)$$

is  $C^r$ -close to  $N$ . Obviously  $\tilde{N}^s \cap U = N^s$ , so if  $s$  is a regular value of  $\pi \circ \varphi$  then  $\tilde{N}^s$  is transverse to  $K \cap U$ . Now cover  $M$  by coordinate neighborhoods  $U_i$  ( $i \in \mathbb{N}$ ) in a locally finite way, that is, such that every compact set intersects only finitely many  $U_i$ . Furthermore take  $U'_i \subset U_i$  such that  $\bar{U}'_i \subset U_i$  and the  $U'_i$  still cover  $M$ . Then the foregoing procedure can be used inductively to produce vectors  $s^i$  and corresponding diffeomorphisms  $T_{s^i}^i$  for  $i \in \mathbb{N}$  such that  $\tilde{N} := (\dots \circ T_{s^k}^k \circ \dots \circ T_{s^1}^1)N$  (locally this is a finite composition) is transverse to  $K$  and  $C^r$ -close to  $N$ .  $\square$

## 8. Locally compact groups and Lie groups

In this section we introduce groups which carry a topology invariant under the group operations. A *topological group* is a group endowed with a topology with respect to which all *left translations*  $L_{g_0} : g \mapsto g_0 g$  and *right translations*  $R_{g_0} : g \mapsto g g_0$  as well as  $g \mapsto g^{-1}$  are homeomorphisms. Familiar examples are  $\mathbb{R}^n$  with the additive structure as well as the circle or, more generally, the  $n$ -torus, where translations are clearly diffeomorphisms, as is  $x \mapsto -x$ . Important other examples are matrix groups, for example,  $GL(n, \mathbb{R})$ ,  $SL(n, \mathbb{R})$ , and so forth, as described after Definition 4.1. A topological group is said to be *locally compact* if every point (or equivalently, the identity) has a neighborhood with compact closure. Such a group possesses a locally finite Borel measure invariant with respect to all right translations, which is unique up to a scalar multiple and called the *right Haar measure*. Similarly, the *left Haar measure* is, up to a scalar multiple, the unique measure invariant with respect to all left translations. These measures are finite if and only if the group is compact. In most interesting cases right invariant Haar measures are also left invariant, for example, when the group is abelian, compact, or, most importantly, a *unimodular linear group*, that is, a closed subgroup of the group  $SL(n, \mathbb{R})$  of all  $n \times n$  matrices with determinant one. In general, groups for which the left and right Haar measures coincide (and naturally are simply called Haar measures) are called *unimodular* and we will restrict our discussion to such groups.

A subgroup  $\Gamma$  of a group  $G$  is called *discrete* if it is closed and all of its points are isolated in the induced topology. In this case the homogeneous space  $G/\Gamma$  (corr.  $\Gamma \backslash G$ ) of orbits of  $R_g$  (corr.  $L_g$ ),  $g \in \Gamma$ , is called the *right* (corr. *left*) *quotient* of  $G$  by  $\Gamma$ . (Unless  $\Gamma$  is a normal subgroup these quotients are not groups.) If either quotient (hence both) is compact in the quotient-topology then  $\Gamma$  is said to be a *uniform* or *cocompact lattice*. It is not difficult to see that for a uniform lattice  $\Gamma$  any right (corr. left) Haar measure on  $G$  projects to a finite Borel measure on the homogeneous space  $G/\Gamma$  (corr.  $\Gamma \backslash G$ ). More generally, a discrete subgroup  $\Gamma$  is called a *lattice* in  $G$  if a right Haar measure projects to a finite measure on  $G/\Gamma$ .

A *non-uniform lattice* is a lattice whose homogeneous space is not compact but still has finite Haar measure. The simplest example, but an extremely important one, especially in number theory, is the subgroup  $SL(2, \mathbb{Z})$  of all matrices with integer elements in the group  $SL(2, \mathbb{R})$  of all  $2 \times 2$  matrices with determinant one; this generalizes to  $SL(n, \mathbb{Z})$  being a non-uniform lattice in  $SL(n, \mathbb{R})$ .

**DEFINITION 8.1.** A *Lie group* is a differentiable manifold with a group structure such that all left and right translations as well as  $g \mapsto g^{-1}$  are diffeomorphisms. A *Lie subgroup* is a subgroup that is a submanifold as well.

Examples of Lie subgroups of  $\mathbb{R}^n$  are linear subspaces as well as integer or rational multiples of a fixed vector and products of these.  $\mathbb{Z}^n$  is a discrete subgroup of  $\mathbb{R}^n$  and  $SL(n, \mathbb{Z})$  (integer matrices with unit determinant) is a discrete subgroup of  $SL(n, \mathbb{R})$ . Note that  $GL(n, \mathbb{Z})$  is *not* a subgroup of  $GL(n, \mathbb{R})$ .

**DEFINITION 8.2.** A *Lie algebra* is a linear space  $g$  with an antisymmetric bilinear operation  $[\cdot, \cdot] : g \times g \rightarrow g$  satisfying the *Jacobi identity*  $[v, [w, z]] + [w, [z, v]] + [z, [v, w]] = 0$ . An *ideal* is a subalgebra  $a \subset g$  such that  $[g, a] \subset a$ .

If  $a, b$  are ideals of  $g$  then so is  $[a, b]$  and thus we have the *derived series*  $g \supset g' := [g, g] \supset g'' := [g', g'] \supset \dots$  of ideals.  $g$  is said to be *abelian* if  $g' = 0$  and *solvable* if the derived series ends at 0 after finitely many steps. A Lie algebra is said to be *semisimple* if 0 is the only solvable ideal, and *simple* if in addition there are no ideals other than  $g$  and 0. The *descending central series* of  $g$  is the sequence  $g^1 := g \supset g^2 := [g, g^1] \supset g^3 := [g, g^2] \supset \dots$  of ideals.  $g$  is said to be *nilpotent* if this series terminates at 0.

The vector fields on any differentiable manifold form an infinite-dimensional Lie algebra with the Lie bracket defined in Definition 4.8. Since left translations are diffeomorphisms they act on vector fields. The Lie bracket is defined on the space  $\mathcal{L}(G)$  of left-invariant vector fields (that is, vector fields invariant under all left translations), so this linear space becomes a Lie algebra called the *Lie algebra of  $G$* .  $\mathcal{L}(G)$  is naturally isomorphic to the tangent space of  $G$  at the identity. Thus its dimension is finite and it coincides with the dimension of  $G$ . Conversely every Lie algebra is the Lie algebra of a unique simply connected Lie group. Important examples of Lie groups are the matrix group  $GL(n, \mathbb{R})$  (nonsingular matrices) and its closed subgroups, such as  $SL(n, \mathbb{R})$  (matrices with unit determinant, the *special linear group*),  $O(\mathbb{R})$  (orthogonal matrices, the *orthogonal group*), and  $SO(n, \mathbb{R})$  (orthogonal matrices with determinant one, the *special orthogonal group*). Here the Lie bracket is given by the commutator  $[A, B] = AB - BA$ . The *exponential map*  $\exp(A) := e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$  defines a map from the tangent space at the identity, that is, the Lie algebra, to the matrix group. This is, in fact, the explicit representation of the abstract exponential map (??) in differential geometry. (This is how the name “exponential” map came about.) On any Lie group a choice of inner product on  $T_{\text{Id}}G$  produces a left-invariant Riemannian metric by left translations. The geodesics through Id for such a Riemannian metric are exactly the one-parameter subgroups of  $G$ . Then  $G$  is a complete differentiable manifold with respect to this metric structure, and the exponential map is always defined on  $T_{\text{Id}}G$ . For matrix groups geodesics through Id are of the form  $t \mapsto e^{tA}$  because these are exactly the one-parameter subgroups. Note that therefore  $\mathcal{L}(G)$  is canonically given as the linear space of matrices obtained from entrywise differentiation of one-parameter subgroups of the matrix group  $G$ . This helps to identify the Lie algebras corresponding to these matrix groups. They are  $\mathcal{L}(GL(n, \mathbb{R})) = \mathfrak{gl}(n, \mathbb{R})$ , the space of all  $n \times n$  matrices;  $\mathcal{L}(SL(n, \mathbb{R})) = \mathfrak{sl}(n, \mathbb{R})$ , the space of traceless matrices (because  $\det e^A = e^{\text{tr} A}$ );  $\mathcal{L}(SO(n, \mathbb{R})) = \mathcal{L}(O(n, \mathbb{R})) = \mathfrak{o}(n, \mathbb{R})$ , the space of skew-symmetric matrices. The *symplectic group*  $Sp(n, \mathbb{R}) \subset GL(2n, \mathbb{R})$  of matrices  $A$  such that  $A^t J A = J$ , where  $J = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}$ , with Id the identity in  $GL(n, \mathbb{R})$ , has Lie algebra  $\mathfrak{sp}(n, \mathbb{R}) = \{A \in \mathfrak{gl}(2n, \mathbb{R}) \mid JA + A^t J = 0 = A^t + A\}$ . The group  $SO(p, q, \mathbb{R})$  of matrices  $A$  such that  $A^t I_{p,q} A = I_{p,q}$ , where  $I_{p,q} = \begin{pmatrix} \text{Id}_p & 0 \\ 0 & \text{Id}_q \end{pmatrix}$ , with  $\text{Id}_k$  the  $k \times k$  identity, has as its Lie algebra the space  $\mathfrak{so}(p, q, \mathbb{R})$  of matrices  $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$  with  $A, C$  skew-symmetric of size  $p, q$ , respectively. An automorphism (that is, diffeomorphic group isomorphism) of a connected Lie group  $G$  is uniquely determined by its differential at the identity. Since a basis in  $T_{\text{Id}}G$  defines a field of bases on  $G$  via left translations there is a left-invariant volume defining the Haar measure. Abelian, compact, discrete, semisimple, and connected nilpotent Lie groups (that is, Lie groups whose Lie algebras are semisimple and nilpotent,

correspondingly) are unimodular. The existence of lattices is not always clear, but  $\mathbb{R}^n$  has plenty, all isomorphic to  $\mathbb{Z}^k$  for some  $k \leq n$ . Lattices in the Heisenberg group came up in ???. The action of the group  $G$  by left translations on a right quotient defines important examples of dynamical systems, where “time”, of course, may be multidimensional. For example, the action of  $\mathbb{R}$  on  $S^1 = \mathbb{R}/\mathbb{Z}$  is the unit-speed flow around the circle.