THE MONOTONICITY OF THE L^p norm

Some of you pointed out to a problem in an old qualifying exam which easily reduces to proving the following:

$$The\ \ norm \quad ||f||_p = (\int |f|^p)^{1/p} \quad is\ \ non-decreasing\ \ in \quad p.$$

Misha Guysinsky in his explanation deduces the statement from a more general inequality which is usually not included into analysis course. I checked proposed solutions for the year's qualifier and there the inequality was assumed so I presume Prof. Pesin discussed a proof in his course.

Here for you elucidation is a proof which is quite straightforward, and, as one should expect, reduces the argument to convexity of a certain function. The point is that the function in question is the logarithm!

Proof. We will simply differentiate the norm with respect to p ans show that the derivative is non-negative (in fact, strictly positive if $f \neq const.$ but we do not need that)

$$\frac{d(\int |f|^p)^{1/p})}{dp} = \left(\int |f|^p\right)^{1/p} \times \frac{d(\frac{\log \int |f|^p}{p})}{dp} =$$

$$\left(\int |f|^p\right)^{1/p} \times \left(\frac{\int(\log|f| \cdot |f|^p)}{p\int |f|^p} - \frac{\log \int |f|^p}{p^2}\right)$$

Since the norm is positive it is sufficient to prove that

$$\frac{\int(\log|f| \cdot |f|^p)}{\int |f|^p} \geq \frac{\log \int |f|^p}{p},$$

or, equivalently,

$$\int(\log|f|^p \cdot |f|^p) \geq \left(\int |f|^p\right) \times \left(\log \int |f|^p\right).$$

Using homogeneity of the norm we may assume without loss of generality that it is equal to one, i.e $\int |f|^p = 1$. Then the inequality follows immediately for the concavity of the logarithm (you may check this by replacing the integral by a finite sum). It also follows that the inequality is strict unless $f = const.$ □