MATH 502: REAL AND COMPLEX ANALYSIS

SPRING 2002

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SECOND MIDTERM EXAMINATION

April 16 , 2004

For a perfect score you should give complete solutions of two problems from each of the two sections.

SECTION 1

1.1. Prove that the set of Lebesgue density points of a measurable set A is a $G_{\delta\sigma}$ set, i.e. a countable union of countable intersections of open sets.

1.2. Prove using only course material (and not any reference to Fourier analysis) that for any measurable set $A \subset [0,1]$

$$\int_A \sin nx \, dx \to 0 \text{ as } n \to \infty.$$

1.3. Let ν_n be a sequence of signed Borel measures on $[0,1]$, $\operatorname{Var}(\nu_n) = 1$. Is it possible that for *every* interval $I \subset [0,1]$ $\nu_n(I) \to 0$?

1.4. Suppose ν is a signed measure on the unit square $I^2 = [0,1] \times [0,1]$ such that for any rectangle $R = [a,b] \times [c,d]$, $\nu(R)$ depends only on $b-a$ and $d-c$.

Prove that ν is proportional to the Lebesgue measure.

SECTION 2

2.1. Let $1 < p < \infty$. Construct a function which belongs to $L^{p-\epsilon}([0,1],\lambda)$ for every $\epsilon > 0$ but not to $L^p([0,1],\lambda)$.

2.2. A function f on $[0,1]$ is *convex* if for any $x, y, t \in [0,1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Prove that any convex function has derivative almost everywhere.

2.3. Prove that any sequence of functions $f_n \in L^p(X,\mu)$ which converges in measure and such that $|f_n(x)| \leq f(x)$, for some function $f \in L^p(X,\mu)$, converges in L^p.

2.4. Prove that for any function $f \in L^1(X,\mu)$ the integral

$$\int_A f d\mu$$

is a continuous function on the metric space of the equivalence classes of measurable sets mod 0 with the distance $d(A,B) = \mu(A\Delta B)$.