MATH 502: REAL AND COMPLEX ANALYSIS

SPRING 2002

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FIRST MIDTERM EXAMINATION

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For a perfect score you should give complete solutions of two problems from each of the two sections.

SECTION 1

1.1. Let X be a compact metric space without isolated points, $C \subset X$ a countable dense subset.

Prove that there is no function on X which is continuous at every point of C and discontinuous at every other point.

1.2. Let X be a compact metric space A Riesz integral on $C(X)$ of the form

$$l(f) = \sum_{i=1}^{k} a_i f(x_i)$$

where $x_1, \dots, x_k \in X$ and $a_1, \dots, a_n > 0$ is called *atomic*. A sequence of Riesz integrals l_n *weakly converges* to l if for any $f \in C(X)$, $l_n(f) \to l(f)$.

Prove that there exists a sequence of atomic Riesz integrals on $C([0,1])$ which weakly converges to the Riemann integral $\int_0^1 f(x)dx$.

1.3. Prove that the uniform limit of Riemann integrable functions is Riemann integrable.

1.4. Show that a bounded function which is the point-wise limit of Riemann integrable functions may not be Riemann integrable.

SECTION 2

2.1. The *support* of a measure μ on a metric space X is the set of all points $x \in X$ such that every open neighborhood of x has positive measure.

Prove that the support of a Borel non-atomic measure on a separable metric space is a perfect set.

2.2. Prove that for any set $A \subset [0,1]$ of positive Lebesgue measure and every $\epsilon > 0$ there exists an interval I such that $\lambda(A \cap I) > (1-\epsilon)\lambda(I)$.

2.3.Prove that the interval $[0,1]$ can be represented as the union of two sets of lower (inner) Lebesgue measure zero,

2.4. Let (X, μ) be a measure space. Consider its *metric algebra*, i.e. the metric space of equivalence classes mod 0 of measurable sets with the distance $d(A,B) = \mu(A \Delta B)$.

Prove that if the metric algebra is compact then the measure μ is atomic.