MATH 502: REAL AND COMPLEX ANALYSIS

SPRING 2004

A.Katok

PROBLEM SET # 9 : March 31

LEBESGUE DENSITY AND ABSOLUTE CONTINUITY

Due on Wednesday 4-7-04

32. Prove the generalization of Lebesgue Density Point Theorem for arbitrary finite Borel measure on an interval.

Hint: The leading special case in a non-atomic measure with full support.

33. Suppose that for a measurable set A and $x \in A$ one can find a sequence $h_n(x) \to 0$ such that $\liminf_{n\to\infty} \frac{h_{n+1}(x)}{h_n(x)} > 0$ and

$$\lim_{n\to\infty} \frac{\lambda(A \cap [x - h_n(x), x + h_n(x)])}{2h_n(x)} = 1.$$

Prove that

$$\lim_{h\to 0} \frac{\lambda(A \cap [x - h, x + h])}{2h} = 1.$$

Here λ is Lebesgue measure.

34. Suppose that μ is a Borel measure on $[0,1]$ such that for a certain constant C and every interval I, $\mu(I) < C\lambda(I)$. Prove that there exists a bounded Lebesgue measurable function ρ such that for every $f \in L^1([0,1], \lambda)$,

$$\int f d\mu = \int \rho f d\lambda.$$

35. Prove that every complex-valued function f on the real line which satisfies Lipschitz condition (i.e $|f(x) - f(y)| \le C|x - y|$ for some $C > 0$ and all $x, y \in \mathbb{R}$) is almost everywhere differentiable.

An extra credit problem

6E. Formulate and prove a counterpart for the Lebesgue Density Point Theorem for the Lebesgue measure in $\mathbb{R}^2$,

Hint: You may use the standard Peano curve and an argument similar to Problem 33.