MATH 502: REAL AND COMPLEX ANALYSIS

SPRING 2004

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PROBLEM SET # 5 : February 26

PRODUCT MEASURES AND LEBESGUE SPACES

Due on Friday, 3-5-04

17. Consider a measurable set $A \subset [0,1]$ of positive Lebesgue measure. Let λ_A be the normalized restriction of Lebesgue measure to A: for a measurable $B \subset A$, $\lambda_A(B) = \frac{\lambda(B)}{\lambda(A)}$. Prove that (A, λ_A) is a Lebesgue space.

Hint: One of possible approaches is to represent A as an F_σ set $\mod 0$.

18. Consider a σ-algebra of subsets of X with non-atomic finite measure μ. Show that for every t, $0 < t < \mu(X)$ there exist uncountably many measurable sets of measure t pairwise different $\mod 0$, i.e. for any two sets A, B from the collection $\mu(A \Delta B) > 0$.

Hint: First construct one such set.

19. Let for $0 < p < 1$, β_p be the measure on $\{0,1\}$ such that $\beta_p(\{0\}) = p$ and $\beta_p(\{0\}) = 1 - p$. Consider the countable product of measures μ_{p_n} on the space Ω_2. Find a necessary and sufficient condition on the sequence p_n for the space Ω_2 with this measure to be a Lebesgue space.

20. Consider a measurable function $f : [0,1] \to \mathbb{R}$ and let $\mathfrak{B}$ be the σ-algebra of the sets of the form $f^{-1}(A)$ where A is a Lebesgue measurable set on the line. Find necessary and sufficient condition for this algebra to be isomorphic $\mod 0$ to Lebesgue σ-algebra.

An extra credit problem

4E. Consider a Borel non-atomic measure μ on the open unit square I^2 such that any open set has positive measure. Prove that there exists a *homeomorphism* $I^2 \to I^2$ such that for any Borel set A, $\mu(A) = \lambda(h(A))$, where λ is Lebesgue measure.

Hint: Try to use an inductive procedure to adjust the measure on finer and finer grids.