MATH 502: REAL AND COMPLEX ANALYSIS

SPRING 2004

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PROBLEM SET #4: February 18

LEBESGUE MEASURE

Due on Wednesday 2-25-04

13. Let μ be a non-atomic (complete) Lebesgue–Stieltjes measure on $[0,1]$. prove that there exists a continuous map $h : [0,1] \to [0,1]$ which is a bijection outside of a set of μ-measure zero and such that for any μ-measurable set $A \subset [0,1]$ the set $h(A)$ is Lebesgue measurable and its Lebesgue measure $\lambda(h(A)) = \mu(A)$.

14. Let f be a bounded Lebesgue measurable function on $[0,1]$. Prove that there exists a sequence of piece-wise constant functions f_n such that $\int_0^1 |f_n(x) - f(x)|dx \to 0$ as $n, \to \infty$

Hint: Use approximation of measurable sets by finite unions of intervals.

15. Construct a Lebesgue measurable set $A \subset [0,1]$ such that both A and its complement intersect any interval $I \subset [0,1]$ by a set of positive Lebesgue measure.

Hint: Use Cantor sets of positive measure as building blocks.

16. Construct a Lebesgue measurable function f on $[0,1]$ such that for any real numbers $a < b$ and any interval $I \subset [0,1]$ the set $x \in I : a < f(x) < b$ has positive Lebesgue measure.

Hint: Use previous problem.

An extra credit problem

3E. Proof existence of a *simple curve* it the plane $\mathbb{R}^2$ of positive plane Lebesgue measure . In other words, find an injective continuous map $f; [0,1] \to \mathbb{R}^2$ such that the set $f([0,1]$ has positive Lebesgue measure in the plane.

Hint: You may first try to construct a continuous map with image of positive measure and then think about modifying it to provide injectivity .