MATH 502: REAL AND COMPLEX ANALYSIS

SPRING 2004

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PROBLEM SET # 3: February 10

LEBESGUE MEASURES AND σ-ALGRBRAS

Due on Wednesday 2-18-04

9. Given a number α, $0 \le \alpha < 1$ consider the transformation T_α of the unit interval $[0,1)$ onto itself:

$$T_\alpha(x) = x + \alpha \text{ if } 0 \le x < 1 - \alpha, \text{ and } x + \alpha - 1 \text{ if } 1 - \alpha \le x < 1.$$

Assume that A is a set such that $T_\alpha^i A \cap T_\alpha^j A = \emptyset$ for any integers $i \ne j$ and $\bigcup_{n \in \mathbb{Z}} T_\alpha^n A = [0,1)$.

Prove that A is not Lebesgue measurable.

10. Consider any closed subset A of the unit interval $[0,1]$.

Prove that there exists a sequence f_n of continuous functions which is fundamental in L^1 (i.e. $\int_0^1 |f_n(x) - f_m(x)|dx \to 0$ as $n, m \to \infty$) which converges point-wise to the characteristic function χ_A.

11. Consider a σ-algebra $\mathfrak{M}$ with a measure μ and let $\mathfrak{M}^\mu$ be the collection of all sets of the form $B = A \cup N_1 \setminus N_2$ where $a \in \mathfrak{M}$ and N_1, N_2 are subsets of null-sets. Define $\tilde{\mu}(B) = \mu(A)$.

Prove that $\mathfrak{M}^\mu$ is a σ-algebra, $\tilde{\mu}$ a measure, and $\mathfrak{M}^\mu$ is saturated with respect to $\tilde{\mu}$.

12. Consider the collection $\mathfrak{Z}$ of all subsets of $\mathbb{Z}$ which are finite union of arithmetic progressions.

Prove that $\mathfrak{Z}$ is an algebra and that there is a measure δ on $\mathfrak{Z}$ which is equal to d^{-1} on any progression with difference d. Prove that δ cannot be extended to a σ-additive measure on the σ-algebra $\mathfrak{B}(\mathbb{Z})$ of all subsets of $\mathbb{Z}$.