

MATH 502: REAL AND COMPLEX ANALYSIS

SPRING 2004

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PROBLEM SET # 1: January 14

NULL-SETS and RIEMANN INTEGRATION

Due on Wednesday 1-21-04

1. Prove that the set of discontinuity points for any function $f : X \rightarrow \mathbb{R}$ on a metric space is the union of a countable number of closed sets.

Hint: Use the notion of *oscillation* of a function at a point.

2. (i) Prove that for any closed null-set X on an interval and for any $\epsilon > 0$ one can find a *finite* set of intervals I_1, \dots, I_n where $n = n(\epsilon)$ such that $\bigcup_{i=1}^n I_i \supset X$ and $\sum_{i=1}^n l(I_i) < \epsilon$.

(ii) generalize this result to a case of Riemann integral on a compact metric space.

3. Prove that for any null-set $X \subset [0, 1]$, $X = \bigcup_{n=1}^{\infty} X_n$, where X_n , $n = 1, 2, \dots$ are closed sets, there exists a Riemann integrable function $f[0, 1] \rightarrow \mathbb{R}$ whose set of discontinuity points coincides with X .

4. Construct an example of a closed nowhere dense subset of the unit interval which is not a null-set.

Hint: Re-interpret the construction of Cantor sets.

An extra credit problem

1E. Prove that every Riemann integrable function on the unit interval is the point-wise limit of a sequence of continuous functions.