

Please write clearly

Name:

## Homework 6, MA 511, Spring 2008

Due on Wednesday, February 27, 2008

Only problems with a \* will be graded

### 1. Linear algebra I

In class we used the following fact (see Perko, Theorem in Section 1.6 “Complex Eigenvalues”):

If  $A$  is real  $2n \times 2n$ -matrix with  $2n$  distinct and non-real eigenvalues, then there is an invertible, real  $2n \times 2n$ -matrix  $P$  such that

$$P^{-1}AP = \text{diag} \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}.$$

Here the right hand side denotes the real  $2n \times 2n$ -matrix

$$\begin{bmatrix} a_1 & -b_1 & 0 & 0 & \cdots & 0 & 0 \\ b_1 & a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_2 & -b_2 & \cdots & 0 & 0 \\ 0 & 0 & b_2 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n & -b_n \\ 0 & 0 & 0 & 0 & \cdots & b_n & a_n \end{bmatrix}.$$

Indeed, we claimed that  $P$  could be realized as follows:

- let the eigenvalues of  $A$  be  $\lambda_j, \bar{\lambda}_j$  with corresponding eigenvectors  $w_j, \bar{w}_j$ , respectively (recall that these must come in complex conjugate pairs since  $A$  is a real matrix);
- let the real and imaginary parts of  $w_j$  be  $u_j \in \mathbb{R}^{2n}$  and  $v_j \in \mathbb{R}^{2n}$ , respectively;
- set  $P := [v_1 | u_1 | \cdots | v_n | u_n]$ .

1. What we showed in class is slightly less than the statement above. Namely, we showed that

$$AP = P \text{diag} \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}.$$

Verify this relation by direct calculation.

2. The issue now arises as to whether  $P$  is necessarily invertible. It is, but we didn't prove that. To see that  $P$  is invertible let's first diagonalize  $A$  over  $\mathbb{C}$ .

- (a) Use well known results from linear algebra to argue why  $A$  is diagonalizable over  $\mathbb{C}$ . What we mean by this is that by setting

$$R := [w_1 | \bar{w}_1 | \cdots | w_n | \bar{w}_n] \in \mathbb{C}^{2n},$$

we have that  $R$  is invertible and that

$$AR = R \operatorname{diag} \begin{bmatrix} \lambda_j & 0 \\ 0 & \bar{\lambda}_j \end{bmatrix}$$

- (b) Let the left (column) eigenvector corresponding to  $\lambda_j$  be  $z_j = \xi_j - i\eta_j$ , where  $\xi_j, \eta_j \in \mathbb{R}^{2n}$ . Then  $\bar{z}_j = \xi_j + i\eta_j$  is the left eigenvector corresponding to  $\bar{\lambda}_j$ . Use the assumption that  $A$  has distinct eigenvalues to conclude that  $\bar{z}_j^T w_k = \delta_{jk}$ . (Here  $^T$  denotes transpose).
- (c) Define the matrix of left eigenvectors by

$$L = \begin{bmatrix} \xi_1^T - i\eta_1^T \\ \xi_1^T + i\eta_1^T \\ \vdots \\ \xi_n^T - i\eta_n^T \\ \xi_n^T + i\eta_n^T \end{bmatrix},$$

and verify the relation

$$LAR = \operatorname{diag} \begin{bmatrix} \lambda_j & 0 \\ 0 & \bar{\lambda}_j \end{bmatrix}.$$

- (d) Use this to conclude that  $z_i^T \bar{w}_i = 0$  for all  $i = 1, \dots, n$ . Finally, define the matrix  $Q$  by

$$Q := [\eta_1 | \xi_1 | \cdots | \eta_n | \xi_n] \in \mathbb{R}^{2n},$$

and use the relations above to show that

$$Q^T P = \frac{1}{2} I_{2n \times 2n},$$

whence  $P$  is invertible.

## \* 2. Exponentials

1. Show that if  $A$  is diagonalizable, then

$$\det e^A = e^{\operatorname{tr} A}.$$

2. Find examples of  $2 \times 2$ -matrices  $A$  and  $B$  such that

$$e^{A+B} \neq e^A e^B.$$

## 3. $2 \times 2$ systems

Consider a real  $2 \times 2$ -matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and let

$$\tau := \operatorname{tr} A, \quad \delta := \det A.$$

1. Show that the eigenvalues of  $A$  are

$$\lambda_{\pm} = \frac{1}{2} \left( \tau \pm \sqrt{\tau^2 - 4\delta} \right).$$

Consider the three mutually exclusive cases where  $A$  has:

- (a) Case I: two distinct, real eigenvalues;
  - (b) Case II: a single real eigenvalue of multiplicity 2;
  - (c) Case III: two distinct, non-real eigenvalues.
2. For each case determine the dimension of the eigenspace(s) (in Case II you may want to subdivide further into three sub-cases: (i):  $a = d$  &  $b = c = 0$ , (ii):  $a = d$  & exactly one of  $b, c$  is zero, (iii):  $a \neq d$  &  $b, c$  both nonzero).
  3. For each case determine all possible phase portraits that the ODE  $\dot{x} = Ax$  can display.

**\* 4. Linear algebra II (only part 4 is graded)**

1. Google “simple proof, jordan decomposition” from a Penn State machine and download the article “A simple Proof of the Jordan Decomposition Theorem for Matrices” by Gohberg & Goldberg from JSTOR. (This is the shortest, yet accessible proof I’ve seen to date).
2. Spend some time looking over the proof. How is it built up? (What’s the general technique of proof they apply?) Do you need any “advanced” algebra to follow it?
3. Their formulation is in terms of (complex) vector spaces and operators on these. Apply their formulation to the case of a complex  $n \times n$ -matrix  $A$  acting on  $\mathbb{C}^n$ . I.e. convince yourself that in this case the theorem amounts to saying that there exists an invertible matrix  $P \in \mathbb{C}^{n \times n}$  such that  $P^{-1}AP$  is block diagonal with *Jordan blocks* along the diagonal. A Jordan block is a  $k \times k$ -matrix of the form

$$J_k(\lambda) = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & \lambda \end{bmatrix}.$$

Here we use the convention that  $J_1(\lambda) = [\lambda]$ , i.e. the scalar  $\lambda$ .

4. \* Conclude from such a representation that  $A$  can be written as a sum  $S + N$  with  $S$  such that  $P^{-1}SP$  is diagonal (with the eigenvalues of  $A$  along its diagonal) and  $N$  is nilpotent.

**\* 5. ODEs, ODEs, ODEs...**

Solve the initial value problem

$$\dot{x} = Ax, \quad x(0) = x_0$$

in the following cases:

- 1.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 1 & 3 & 2 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

3.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

4.

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

**\* 6. True or false?**

Assume that  $A \in \mathbb{R}^{2 \times 2}$  has two eigenvalues with strictly negative real parts. Let  $x(t)$  be a solution of  $\dot{x} = Ax$ . Prove or disprove (by counterexample) the following assertions:

1.  $|x(t)|$  decreases as  $t$  increases;
2. there exists a constant  $a > 0$  such that  $|x(t)| \leq e^{-at}$  for all  $t$ .

**7. Second order equation**

Assume  $a, b$  are strictly negative real constants. Without explicitly finding the solution, determine the long time behavior of the solution to

$$\ddot{x} + a\dot{x} + bx = 0, \quad x(0) = x_0.$$

**\* 8. Liapunov function**

Suppose each eigenvalue of the real  $n \times n$ -matrix  $A$  has strictly negative real part and define the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$V(x) := \int_0^\infty |e^{tA}x|^2 dt.$$

Why is this integral well-defined? Show that if  $x = x(t)$  is a solution of  $\dot{x} = Ax$ , then  $t \mapsto V(x(t))$  is a monotonically decreasing function of time.