Please write clearly

Name:

Homework 2, MA 511, Spring 2008

Due on Wednesday, January 30, 2008

Only problems with a * will be graded

* 1. Scalar autonomous equations

For this problem x denotes a *scalar* function of time, i.e. $x(t) \in \mathbb{R}$.

1. Consider the second order autonomous ODE

$$\ddot{x} = \phi(x)\,. \tag{1}$$

Let Φ be a primitive of ϕ, i.e. $\Phi'(z) = \phi(z)$. Show that any solution $x(t)$ of (1) satisfies

$$\frac{(\dot{x})^2}{2} - \Phi(x(t)) \equiv \text{constant}\,.$$

2. Next consider the initial value problem

$$\dot{x} = f(x)\,, \qquad x(t_0) = x_0\,. \tag{2}$$

Assume that the function f is continuous and defined everywhere on $\mathbb{R}$ and does not vanish. For concreteness let's assume that $f(z) > 0$ for all z. Show that any solution of (2) must satisfy

$$F(x(t)) = t - t_0$$

where F is defined by

$$F(z) := \int_{x_0}^{z} \frac{d\xi}{f(\xi)}\,.$$

Argue that this last relation determines $x(t)$ uniquely.

3. Notice that the argument in part 2. shows that the initial value problem (2) has a *unique* solution. Reconcile this with the standard *non-uniqueness* example from Homework 1.

* 2. Extensions

1. Consider the initial value problem

$$\dot{x} = \frac{1}{2x}\,, \qquad x(1) = 1\,. \tag{3}$$

   Identify the maximal interval of existence of the solution.

2. Solve the following initial value problems, find the maximal interval of existence (α, β) of the solution, and determine the behavior as $t \to \alpha+$, $\beta-$.

   (a)

$$\begin{aligned} \dot{x}_1 &= x_1^2 & x_1(0) &= 1 \\ \dot{x}_2 &= x_2 + \tfrac{1}{x_1} & x_2(0) &= 1 \end{aligned}$$

   (b)

$$\begin{aligned} \dot{x}_1 &= \tfrac{1}{2x_1} & x_1(0) &= 1 \\ \dot{x}_2 &= x_2^2 & x_2(0) &= 1 \end{aligned}$$

   (c)

$$\begin{aligned} \dot{x}_1 &= -\tfrac{x_2}{x_3^2} & x_1(1/\pi) &= 0 \\ \dot{x}_2 &= \tfrac{x_1}{x_3^2} & x_2(1/\pi) &= 1 \\ \dot{x}_3 &= 1 & x_3(1/\pi) &= 1/\pi\,. \end{aligned}$$

* 3. Comparison of solutions for scalar first order equations

Consider two autonomous, first order, scalar ODEs:

$$\dot{x} = f(x)\,, \qquad \dot{y} = g(y)\,.$$

Assume that f, g are continuous, $g(x) > f(x)$ for all x, and that the starting values satisfy $y(0) \geq x(0)$. For simplicity let's say that the solutions $x(t)$ and $y(t)$ exist for all $t \geq 0$. Show that $y(t) \geq x(t)$ for all t. (Hint: argue by contradiction).

4. Linear algebra A $k \times k$ matrix of the form

$$J_k(\bar{\lambda}) = \begin{bmatrix} \bar{\lambda} & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \bar{\lambda} & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \bar{\lambda} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & \bar{\lambda} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \bar{\lambda} \end{bmatrix}$$

is called a *Jordan block.*

1. Show that $J_k(\bar{\lambda})$ has only one eigenvalue - what is it? What is its algebraic multiplicity (i.e. its multiplicity as a root of the characteristic equation $\det(J_k(\bar{\lambda}) - \lambda I)$, I being the $k \times k$ identity matrix)? What is its geometric multiplicity (i.e. the dimension of its eigenspace)?

2. Suppose a $k \times k$-matrix A is similar to $J_k(\bar{\lambda})$, i.e. there exists an invertible $k \times k$-matrix P such that

   $$P^{-1}AP = J_k(\bar{\lambda})\,.$$

   Let the column vectors of P be $v_1, \ldots, v_k$:

   $$P := \left[\, v_1 \,\middle|\, \cdots \,\middle|\, v_k \,\right].$$

   Check that we then have

   $$\begin{aligned} Av_1 &= \bar{\lambda} v_1\,, \\ Av_j &= \bar{\lambda} v_j + v_{j-1} \qquad \text{for } j = 1, \ldots, k-1. \end{aligned}$$

5. Uniqueness and log-Lipschitz equations

Consider the autonomous Cauchy problem

$$\dot{x}(t) = f(x)\,, \qquad x(0) = x_0\,. \tag{4}$$

For simplicity assume x is a single real-valued function. As we showed in class, Lipschitz continuity of $f(x)$ near x_0 guarantees *uniqueness* of a solution. From the example $\dot{x} = x^{\frac{1}{3}}$, $x(0) = 0$ we see that Hölder continuity of f is *not* sufficient to guarantee uniqueness. A reasonable guess would be that if $|f'(x)| \to \infty$ as $x \to x_0$, then there are more than one solution to (4). The purpose of this exercise is to show, maybe surprisingly, that this is not the case. A particular case is provided by the so-called log–Lipschitz functions.

Definition 1. *Let $I \subset \mathbb{R}$ be an open interval. A map $f : I \to \mathbb{R}$ is said to be* log-Lipschitz *continuous at $a \in I$ provided there is a finite constant K such that*

$$\sup_{0<|x-a|<\frac{1}{2}} \frac{|f(x) - f(a)|}{|x-y| \cdot \big|\log|x-a|\big|} \leq K\,. \tag{5}$$

1. Verify that if f is Lipschitz continuous at a then it is also log-Lipschitz continuous at a.
2. Verify that the function $f(x) = x \log x$ (defined to be 0 at $x = 0$) is log-Lipschitz continuous at $x = 0$. Is f is Lipschitz continuous at $x = 0$?
3. Identify one (obvious) solution to $\dot{x} = x \log x\,,\ x(0) = 0$.
4. Try solving the same problem by observing that the equation is separable and that
$$\frac{d}{dx} \log(\log x) = \frac{1}{x \log x}\,.$$
5. Now suppose $f(x)$ is log-Lipschitz continuous at all points in I (with a common constant K) and let's denote the solution of (4) by $x = x(t; x_0)$. We consider two trajectories $x(t) = x(t; x_0)$ and $y(t) = x(t; y_0)$ with $|x_0 - y_0| < \frac{1}{4}$, say, and we consider times t sufficiently near $t = 0$ so as to guarantee that $|x(t) - y(t)| < \frac{1}{2}$. Then define
$$\delta(t) := |x(t) - y(t)|\,,$$
and assume $\delta(t)$ is differentiable at all times t^1 of interest. Show that
$$\dot{\delta}(t) \leq K \cdot \delta(t) \cdot \big|\log \delta(t)\big|\,.$$
Integrate this differential inequality (e.g. by introducing $\mu(t) := \log \delta(t)$) to obtain that
$$\delta(t) \leq \sqrt{\delta(0)}\,, \qquad \text{for sufficiently small times } t.$$
Conclude uniqueness of trajectories.

*6. An example of successive approximations

Consider the method of successive approximation for the problem:

$$\dot{x} = f(t, x)\,, \qquad x(0) = 0\,,$$

where we assume that $f(t, x)$ is a continuous function such that

$$f(s, 0) = -2s\,, \qquad f(s, -s^2) = 0\,, \qquad \text{for all } s \in \mathbb{R}.$$

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[1] It is differentiable *almost everywhere*

1. With the initial guess $x_0(t) \equiv 0$, calculate a few iterates according to

$$x_{n+1}(t) = \int_0^t f(s, x_n(s))\, ds\,.$$

2. Show that $x_{2n}(t) \equiv 0$ and that $x_{2n+1}(t) = -t^2$ for all n.

3. Clearly, the method of successive approximations cannot work in this case. Still, we used this method to prove existence (and uniqueness) for ODEs. Explain why there is no contradiction here. (Hint: try to provide a simple example of an f with the given properties - draw the graph of f near $(t, x) = (0, 0)$. What can you say about the behavior of f near $(0, 0)$?)