Please write clearly

Homework 12, MA 511, Spring 2008

Due on Friday, April 25, 2008
Only problems with a * will be graded

* 1. Smoothness of rescalings and distance functions

1. Given a C^1-function $f : E^{\text{open}} \subset \mathbb{R} \to \mathbb{R}$, show that the function
$$\tilde{f}(x) := \frac{f(x)}{1 + |f(x)|}$$
is also C^1-smooth on E. (Hint: consider separately points where f vanishes.)
2. Prove the same when $f : E^{\text{open}} \subset \mathbb{R}^n \to \mathbb{R}$.
3. Given a set $F \subset \mathbb{R}^n$, show that the distance function
$$\delta(x) := \operatorname{dist}(x, F) \equiv \inf_{y \in F} |x - y|$$
is Lipschitz continuous with Lipschitz constant equal to 1.
4. Give an example which shows that δ is not C^1-smooth: neither on $\mathbb{R}^n$ nor when restricted to F^c, the complement of F. (Hint: $n = 1$.)
5. Google and memorize Rademacher's theorem on almost everywhere differentiability of Lipschitz functions.

*2. More on Lyapunov functions for linear systems

Consider a linear system
$$\dot{x} = Ax\,,$$
where $A \in \mathbb{R}^{n\times n}$ is such that $\sigma(A) \subset \mathbb{C}_-$ (i.e. each eigenvalue of A has strictly negative real part). Define the (Lyapunov) function
$$V(x) := \int_0^\infty |e^{tA}x|^2\,dt\,. \tag{1}$$

1. Show that V is a quadratic form in x: there is an $n \times n$-matrix Q such that
$$V(x) = x^T Q x\,.$$
Identify Q explicitly. (It will involve an integral; in particular argue why the integral is convergent.)

2. Show that Q satisfies the matrix equation

$$A^T Q + QA = -I_{n\times n}\,.$$

Next consider another scalar function defined along solutions of the same linear system: for $y \in \mathbb{R}^n$ we let $L(y)$ denote the arc length of the trajectory of the solution $x(t)$ that starts at y. That is,

$$L(y) := \int_0^\infty \left|\tfrac{d}{ds} e^{sA} y\right| ds\,. \tag{2}$$

3. Give a detailed argument for why $L(y)$ is well-defined (i.e. the integral is convergent).

4. Calculate directly from the definition that

$$\frac{d}{dt} L(x(t)) = -|Ax(t)|\,.$$

   Use this to show that L is a strict Lyapunov function for the ODE $\dot{x} = Ax$. (Hint: evaluate at $t = 0$.)

5. Recall from a previous homework how we used the Lyapunov function V to prove a stability result for the perturbed system $\dot{x} = Ax + g(x)$, where $g(x) \sim o(x)$ as $x \to 0$. What obstruction (if any) is there to using the Lyapunov function L in the same way?

3. Planetary trajectories are conic sections (approximately)

Consider the potential $V(x) = -1/|x|$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ for which the corresponding Newton's equation is

$$\ddot{x} = -\frac{x}{|x|^3}\,.$$

The associated Hamiltonian is

$$H(x, v) := \frac{1}{2}|v|^2 - \frac{1}{|x|}\,, \tag{3}$$

where $v = (v_1, v_2, v_3) = \dot{x}$. From an earlier homework we know that also $\omega := x \times v$ is constant in time. As $v \perp \omega$ it follows that the motion of the particle takes place in a fixed plane. By a suitable choice of coordinates we can assume that this plane is the (x_1, x_2)-plane, i.e. $x_3 = v_3 \equiv 0$. We will assume that

$$\omega := x_1 v_2 - x_2 v_1 \neq 0\,.$$

1. What does this assumption mean in terms of the trajectory of the particle?

2. We introduce standard polar coordinates (r, θ) in the plane. Show that

$$\frac{d}{dt}\left(\frac{x_1}{r}\right) = \omega \dot{v}_2\,.$$

3. Use this (and a similar calculation for x_2/r) to conclude that there are constants C, D such that

$$\omega^2 = Cx_1 + Dx_2 + r\,.$$

4. Recall that a conic section with eccentricity e is described in polar coordinates by

   $$r(\theta) = \frac{\ell}{1 + e\cos(\theta - \theta_0)}$$

   where ℓ, θ_0 are constants[1] and $e \in (0,1)$ for ellipses, $e = 1$ for parabolas, and $e > 1$ for hyperbolas. Show that the trajectory of x is a conic section.

* 4. Globally defined solutions

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz: there is a constant L such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}^n$. Let $\phi(t, x)$ denote the flow of the associated ODE $\dot{x} = f(x)$.
Prove that every trajectory $\phi(t, x_0)$ is defined for all $t \in \mathbb{R}$.

(Hint: Use Grönwall's Lemma to estimate $|\phi(t, x_0) - x_0|$ and argue that if $\phi(t, x_0)$ were not defined for all $t > 0$, say, then the forward trajectory would be contained in a compact set.)

* 5. Attractors

Identify the attracting set for the system

$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 - y^2 - z^2) && (4)\\ \dot{y} &= x + y(1 - x^2 - y^2 - z^2) && (5)\\ \dot{z} &= 0\,. && (6) \end{aligned}$$

* 6. The Lorenz equations

The Lorenz system is given by:

$$\begin{aligned} \dot{x} &= \sigma(y - x) && (7)\\ \dot{y} &= rx - y - xz && (8)\\ \dot{z} &= xy - bz\,, && (9)\\ & && (10) \end{aligned}$$

where σ, r, b are positive parameters. Verify the following:

1. If $(x(t), y(t), z(t))$ is a solution, then so is $(-x(t), -y(t), z(t))$ (two first components negated).

2. The z-axis is invariant and consists of three trajectories.

3. Assume $r \in (0,1)$; then the origin is uniformly stable in the sense of Lyapunov. (Hint: look for a quadratic Lyapunov function.)

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[1] The constant ℓ is called the semi-latus rectum(!)

4. For $r > 1$ there are three equilibrium points: at the origin, and at $\big(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\big)$. For $r > 1$ there is a one dimensional unstable manifold at the origin.

7. Lagrange's equation for conservative systems

Recall that the Hamiltonian associated with a particle of (constant) mass m in a conservative force field with potential V is given by

$$H(q,p) := \frac{1}{2m}|p|^2 + V(q)\,,$$

and Newton's 2nd law takes the form

$$\dot{p} = -\nabla V(q)\,.$$

Now introduce the *Lagrangian* L by

$$L(q,p) := \frac{1}{2m}|p|^2 - V(q)\,,$$

1. Show that Newton's 2nd law may be written as

   $$\frac{d}{dt}(\nabla_p L) = (\nabla_q L)\,. \tag{11}$$

   This may seem like a trivial reformulation. However, Lagrange's motivation for introducing his formalism was that the direct application of Newton's second law "$F = ma$" may be cumbersome to implement for mechanical systems with constraints (e.g. a double pendulum). Such constraints can often be expressed by requiring that the configuration of the system lies in some manifold $M \subset \mathbb{R}^n$. A key property of Lagrange's equation (11) is its *invariance* under changes of coordinates.

2. Suppose $\tilde{q}$ is another (generalized) position, and let the change of coordinates be given by $q = f(\tilde{q})$. Then we have

   $$p = Df(\tilde{q})\tilde{p}\,,$$

   where we have defined the (generalized) velocities $p := \dot{q}$ and $\tilde{p} := \dot{\tilde{q}}$. Show that Lagrange's equation takes the same form as (11) when written in the $(\tilde{q}, \tilde{p})$ coordinates.

8. Gradient systems

Let $V : E^{\text{open}} \subset \mathbb{R}^N \to \mathbb{R}$ be a of class $C^2(E)$. Then the ODE system

$$\dot{x} = -\nabla V(x) \tag{12}$$

is called a *gradient system* on E (associated with V).

1. Verify that the rest points of (12) are exactly the critical points of V, and explain why the system obtained by linearizing (12) at any rest point has only real eigenvalues.

2. Consider the case $N = 2n$ and let $x = (q, p)$. Assume that we are given a Hamiltonian system

$$\begin{aligned} \dot{q} &= \nabla_p H(q, p)\,, & (13) \\ \dot{p} &= -\nabla_q H(q, p)\,. & (14) \end{aligned}$$

   Show that the gradient system associated with $V(q, p) := -H(q, p)$ has solution trajectories that are normal to the level sets of the Hamiltonian H.

* 9. Limit cycles

Consider the system

$$\begin{aligned} \dot{x} &= -y + x(x^2 + y^2) \sin\left(\tfrac{1}{\sqrt{x^2+y^2}}\right) & (15) \\ \dot{y} &= x + y(x^2 + y^2) \sin\left(\tfrac{1}{\sqrt{x^2+y^2}}\right), & (16) \end{aligned}$$

and with $\dot{x} = \dot{y} = 0$ at $(0, 0)$. Show that this is a C^1 system on $\mathbb{R}^2$ and with the property that the origin is a an equilibrium point, while each circle Γ_n with radius $\frac{1}{n\pi}$ is a limit cycle. Furthermore, these limit cycles accumulate at the origin, Γ_{2n} is stable, while Γ_{2n+1} is unstable. (Hint: polar coordinates.)