Please write clearly

Homework 10, MA 511, Spring 2008

Due on Friday, April 11, 2008

Only problems with a * will be graded

* 1. Application I: Heat conduction again

Consider n identical balls connected by heat-conducting rods in such a way that each pair of balls is directly connected through a single rod. Let x_i denote the temperature of the ith ball, and let $a_{ij} = a_{ji}$ be the (constant and positive) heat conductivity of the rod connecting the ith and jth balls. As usual we assume that the heat flows through the rod connecting ball i and ball j at the rate $a_{ij} \cdot (x_i - x_j)$, where x_i is the temperature of the ith ball, and that no heat is lost to the surrounding medium.

1. Recall the system $\dot{x} = Ax$ that governs the time evolution. Show that A is negative semi-definite by explicitly calculating $x^T Ax$ (using the expressions for A_{ij} in terms of the a_{ij}). From this, what can you say about the eigenvalues of A?

2. Let ℓ denote the one-dimensional subspace of $\mathbb{R}^n$ spanned by the vector $\mathbf{1} := (1, \ldots, 1)^T$, and let $\mathcal{X} = \ell^\perp$, i.e., the $(n-1)$-dimensional orthogonal complement of ℓ. Use the calculation in the first part to show that A restricted to $\mathcal{X}$ is *strictly* negative definite. Use this to list the stable, unstable, and center subspaces of the original system. Draw a picture of the situation in the case $n = 3$.

3. Use your findings to briefly describe the dynamics of this linear system (large time behavior, limiting state expressed in terms of initial data etc.).

* 2. Stable and unstable manifolds

In special cases one may be able to find an explicit expression for the general solution of a nonlinear ODE system. If the system consists of n equations, the solution will depend on n constants. For example these could be just the initial coordinates of the solution. Given the general solution one can then identify the stable and unstable manifolds by studying how the asymptotic behavior of the solution depends on the n constants. Here is an example:

$$\begin{aligned}\dot{x} &= -x \\ \dot{y} &= -y + x^2 \\ \dot{z} &= z + y^2 \,.\end{aligned}$$

1. Linearize the system at the origin (the only rest point of the system), and identify the stable and unstable subspaces for the linearized system.

2. Solve the nonlinear system explicitly with initial data $(x(0), y(0), z(0)) = (a, b, c)$.

3. Use this to show that the stable manifold at the origin is given by the graph:

$$\mathcal{S} = \left\{ (x, y, z) \,|\, z = -\tfrac{1}{3}y^2 - \tfrac{1}{6}x^2 y - \tfrac{1}{30}x^4 \right\}$$

4. Also show that the unstable manifold at the origin is the z-axis.

5. Verify that the stable and unstable manifolds are indeed tangent to the stable and unstable subspaces, respectively.

* 3. Stability analysis via polar coordinates[1]

Assume that the origin is a non-hyperbolic rest point for the system $\dot{x} = f(x)$. In this case the linearization at the origin, $\dot{x} = Ax$, $A = Df(0)$, may not predict the correct stability properties of the nonlinear system. Consider the system

$$\begin{aligned} \dot{x} &= -y - ax\sqrt{x^2 + y^2} \\ \dot{y} &= x - ay\sqrt{x^2 + y^2}\,, \end{aligned}$$

where a is a constant, and consider the rest point at the origin.

1. Show that the origin is a center for the linearized system.

2. One way to analyze the nonlinear stability of the zero solution is to change to polar coordinates. Derive and explicitly solve the corresponding system in polar coordinates.

3. Show that if $a \neq 0$ then the origin is *not* a center for the nonlinear system.

4. (Extra, open problem) Another approach would be to search for a Lyapunov function. Can you find one in the case that $a > 0$?

* 4. Phase portrait via blowup in polar coordinates[2]

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= x^2 - 2xy \\ \dot{y} &= y^2 - 2xy\,, \end{aligned}$$

and the rest point at the origin.

1. Linearize at the origin and conclude that no information about the phase portrait is available at the linear level.

[1] After Verhulst "Nonlinear Differential Equations and Dynamical Systems".
[2] After Chicone "ODEs with Applications" 2nd ed.

2. Write out the corresponding system in polar coordinates.

3. Argue why the phase portrait of this transformed system is the same as that of the "desingularized" system

$$\begin{aligned} \dot{r} &= r(\cos^3\theta - 2\cos^2\theta\sin\theta - 2\cos\theta\sin^2\theta + \sin^3\theta) \\ \dot{\theta} &= 3\cos\theta\sin\theta(\sin\theta - \cos\theta)\,, \end{aligned}$$

4. Linearize *this* system about the rest points $(r, \theta) = (0, \theta)$ for $\theta = k\pi/4$, $k = 0, 1, 2, 4, 5, 6$.

5. Deduce that all of these are hyperbolic rest points for the (r, θ)-system and use this to sketch the phase portrait of the original system about the origin.

***5. Finding a Lyapunov function**

Consider the system

$$\begin{aligned} \dot{x} &= -2y + yz \\ \dot{y} &= x - xz \\ \dot{z} &= xy\,. \end{aligned}$$

1. Linearize at the origin and show that this is a degenerate rest point (i.e. the linearization has zero as an eigenvalue).

2. In order to analyze the stability of the zero solution, try searching for a Lyapunov function in the form of a positive definite quadratic form. That is, make the ansatz that

$$V(x, y, z) = ax^2 + by^2 + cz^2\,, \qquad \text{with } a,\ b,\ c > 0,$$

and determine a, b, and c such that V becomes a Lyapunov function.

3. Apply the same technique to find a strict Lyapunov function for the system

$$\begin{aligned} \dot{x} &= -2y + yz - x^3 \\ \dot{y} &= x - xz - y^3 \\ \dot{z} &= xy - z^3\,. \end{aligned}$$

Conclude that the origin is an asymptotically stable rest point. What type of rest point is the origin for the linearized system? Comment.

*6. À propos the Hartman-Grobman theorem[3]

Carefully read the statement of the Hartman-Grobman theorem. In connection with this theorem, let's observe that a given homeomorphism acting on phase space may well change the phase portrait of an ODE in a significant manner. Consider the map $H : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$H(u,v) = (u\cos s - v\sin s, u\sin s + v\cos s)\,, \qquad \text{where } s = -\tfrac{1}{2}\ln(u^2+v^2)\,.$$

1. Assume that (u,v) solves the system

$$\dot{u} = -u\,, \qquad \dot{v} = -v\,,$$

   and set $(x,y) := H(u,v)$. Show that x and y solve the system

$$\dot{x} = -x - y\,, \qquad \dot{y} = x - y\,.$$

2. Verify that H is a homeomorphism (provided we define $H(0,0) = (0,0)$). Show that the origin is a (stable) node for the (u,v)-system, while it is a (stable) focus for the (x,y)-system. Thus, in the Hartman-Grobman theorem one *could*, presumably, have a situation where the "linearizing homeomorphism" H distorts the phase portrait in a way similar to the example above.

3. Verify that the H above is *not* a *diffeomorphism*[4].

7. Decomposition of flows[5]

Consider an ODE system

$$\dot{x} = f(x)\,, \tag{1}$$

where $f : \Omega^{\text{open}} \subset \mathbb{R}^n \to \mathbb{R}^n$ is of class C^2 and with a critical point at the origin: $f(0) = 0$. We set $A := Df(0)$ and $g(x) := f(x) - Ax$ and rewrite the system as

$$\dot{x} = Ax + g(x)\,, \tag{2}$$

where $g(0) = 0$ and $Dg(0) = 0$. We let V^c, V^s, V^u denote respectively the center, stable, and unstable subspaces associated with the matrix A. The corresponding projections onto these subspaces are denoted π^c, π^s, and π^u. In particular we have

$$x = \pi^c x + \pi^s x + \pi^u x\,, \qquad \forall\, x \in \mathbb{R}^n\,.$$

[3] After lecture notes by Hanche-Olsen

[4] It's tempting to try and prove a version of the Hartman-Grobman theorem where the map H is a diffeomorphism. However, Hartman himself showed that there are systems (with analytic right-hand sides) for which there is no diffeomorphim H satisfying the conclusion of the Hartman-Grobman theorem. An example of his is the system

$$\dot{x} = ax\,, \qquad \dot{y} = (a-b)y + cxz\,, \qquad \dot{z} = -bz\,, \qquad \text{with } a > b > 0 \text{ and } c \neq 0.$$

[5] After notes of A. Bressan

1. Derive by "variation of constants" that any solution satisfies

$$x(t) = e^{(t-t_0)A}x(t_0) + \int_{t_0}^{t} e^{(t-\tau)A} g(x(\tau))\, d\tau \,.$$

2. Use this to show that for any triple of "starting times" t_c, t_s, t_u there are "starting points" $x_c \in V^c$, $x_s \in V^s$, $x_u \in V^u$ such that the formula above takes the form

$$\begin{aligned} x(t) \quad = \quad & \pi^c\Big(e^{(t-t_c)A}x_c + \int_{t_c}^{t} e^{(t-\tau)A} g(x(\tau))\, d\tau\Big) \\ & + \pi^s\Big(e^{(t-t_s)A}x_s + \int_{t_s}^{t} e^{(t-\tau)A} g(x(\tau))\, d\tau\Big) \\ & + \pi^u\Big(e^{(t-t_u)A}x_u + \int_{t_u}^{t} e^{(t-\tau)A} g(x(\tau))\, d\tau\Big) \,. \end{aligned} \tag{3}$$

8. Application II: Predator-pery model as a Hamiltonian system

Consider the predator-prey model

$$\begin{aligned} \dot{x} \quad &= \quad ax - bxy \\ \dot{y} \quad &= \quad -cy + dxy \,, \end{aligned}$$

where a, b, c, d are positive constants, and $x(t)$, $y(t)$ model populations of prey and predators, respectively. Verify that, by setting $x := e^p$ and $y := e^q$, the system becomes a planar Hamiltonian system. That is, show that there is a function $(p, q) \mapsto H(p, q) \in \mathbb{R}$ such that the system in (p, q)-coordinates takes the form

$$\begin{aligned} \dot{q} \quad &= \quad \partial_p H(p, q) \\ \dot{p} \quad &= \quad -\partial_q H(p, q) \,. \end{aligned}$$