Full Reflection of Stationary Sets Below \( \aleph_\omega \)

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Abstract

It is consistent that for every \( n \geq 2 \), every stationary subset of \( \omega_n \) consisting of ordinals of cofinality \( \omega_k \) where \( k = 0 \) or \( k \leq n - 3 \) reflects fully in the set of ordinals of cofinality \( \omega_{n-1} \). We also show that this result is best possible.

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1. Introduction.

A stationary subset $S$ of a regular uncountable cardinal $\kappa$ reflects at $\gamma < \kappa$ if $S \cap \gamma$ is a stationary subset of $\gamma$. For stationary sets $S, A \subseteq \kappa$ let

$$S < A \text{ if } S \text{ reflects at almost all } \alpha \in A$$

where “almost all” means modulo the closed unbounded filter on $\kappa$, i.e., with the exception of a nonstationary set of $\alpha$’s. If $S < A$ we say that $S$ reflects fully in $A$. The trace of $S, Tr(S)$, is the set of all $\gamma < \kappa$ at which $S$ reflects. The relation $<$ is well-founded [1], and $o(S)$, the order of $S$, is the rank of $S$ in this well-founded relation.

In this paper we investigate the question which stationary subsets of $\omega_n$ reflect fully in which stationary sets; in other words the structure of the well-founded relation $<$. Clearly, $o(S) < o(A)$ is a necessary condition for $S < A$, and moreover, a set $S \subseteq \omega_n$ has order $k$ just in case it has a stationary intersection with the set

$$S^n_k = \{ \alpha < \omega_n : cf \alpha = \omega_k \}.$$ 

Thus the problem reduces to the investigation of full reflection of stationary subsets of $S^n_k$ in stationary subsets of $S^n_m$ for $k < m < n$.

The problem for $n = 2$ is solved completely in Magidor’s paper [2]: It is consistent that every stationary $S \subseteq S^2_0$ reflects fully in $S^2_1$. The problem for $n > 2$ is more complicated. It is tempting to try the obvious generalization, namely $S < A$ whenever $o(S) < o(A)$, but this is provably false:

**Proposition 1.1.** There exist stationary sets $S \subseteq S^3_0$ and $A \subseteq S^3_1$ such that $S$ does not reflect at any $\gamma \in A$.

**Proof.** Let $S_i, i < \omega_2$, be any family of pairwise disjoint subsets of $S^3_0$, and let $\langle C_\gamma : \gamma \in S^3_1 \rangle$ be such that each $C_\gamma$ is a closed unbounded subset of $\gamma$ of order type $\omega_1$. Clearly, at most $\aleph_1$ of the sets $S_i$ can meet each $C_\gamma$, and so for each $\gamma$ there is $i(\gamma) < \omega_2$ such that $C_\gamma \cap S_i = \emptyset$ for all $i \geq i(\gamma)$.

There is $i < \omega_2$ such that $i(\gamma) = i$ for a stationary set of $\gamma$’s. Let $A \subseteq S^3_1$ be this stationary set and let $S = S_i$. Then $S \cap C_\gamma = \emptyset$ for all $\gamma \in A$ and so $S \cap \gamma$ is nonstationary. Hence $S$ does not reflect at any $\gamma \in A$. $\blacksquare$
There is of course nothing special in the proof about $\aleph_3$ (or about $\aleph_1$) and so we have
the following generalization:

**Proposition 1.2.** Let $k < m < n - 1$. There exist stationary sets $S \subseteq S^n_k$ and $A \subseteq S^n_m$ such that $S$ does not reflect at any $\gamma \in A$. □

Consequently, if $n > 2$ then full reflection in $S^n_m$ is possible only if $m = n - 1$. This motivates our Main Theorem.

1.3 **Main Theorem.** Let $\kappa_2 < \kappa_3 < \cdots < \kappa_n < \cdots$ be a sequence of supercompact cardinals. There is a generic extension $V[G]$ in which $\kappa_n = \aleph_n$ for all $n \geq 2$, and such that

(a) every stationary subset of $S^n_0$ reflects fully in $S^n_1$, and

(b) for every $n \geq 3$, every stationary subset of $S^n_k$ for all $k = 0, \cdots, n - 3$, reflects fully in $S^n_{n-1}$.

We will show that the result of the Main Theorem is best possible. But first we prove a corollary:

1.4 **Corollary.** In the model of the Main Theorem we have for all $n \geq 2$ and all $m, 0 < m < n$:

(a) Any $\aleph_m$ stationary subsets of $S^n_0$ reflect simultaneously at some $\gamma \in S^n_m$.

(b) For every $k \leq m - 2$, any $\aleph_m$ stationary subsets of $S^n_k$ reflect simultaneously at some $\gamma \in S^n_m$.

**Proof.** Let us prove (a) as (b) is similar. Let $m < n$ and let $S_\xi, \xi < \omega_m$, be stationary subsets of $S^n_0$. First, each $S_\xi$ reflects fully in $S^n_{n-1}$ and so there exist club sets $C_\xi, \xi < \omega_m$, such that each $S_\xi$ reflects at all $\alpha \in C_\xi \cap S^n_{n-1}$. As the club filter is $\omega_n$ - complete, there exists an $\alpha \in S^n_{n-1}$ such that $S_\xi \cap \alpha$ is stationary, for all $\xi < \omega_m$. Next we apply full reflection of subsets of $S^n_{n-1}$ in $S^n_{n-2}$ (to the ordinal $\alpha$ of cofinality $\omega_{n-1}$ rather than to $\omega_{n-1}$ itself) and the $\omega_{n-1}$ - completeness of the club filter on $\omega_{n-1}$, to find $\beta \in S^n_{n-2}$ such that $S_\xi \cap \beta$ is stationary for all $\xi < \omega_m$. This way we continue until we find a $\gamma \in S^n_m$ such that every $S_\xi \cap \gamma$ is stationary. □

Note that the amount of simultaneous reflection in 1.4 is best possible:
1.5 **Proposition.** If \( cf \gamma = \aleph_m \) and if \( S_\xi, \xi < \omega_{m+1} \), are disjoint stationary sets then some \( S_\xi \) does not reflect at \( \gamma \).

**Proof.** \( \gamma \) has a club subset of size \( \aleph_m \), and it can only meet \( \aleph_m \) of the sets \( S_\xi \cap \gamma \). □

By Corollary 1.4, the model of the Main Theorem has the property that whenever \( 2 \leq m < n \), every stationary subset of \( S_k^m \) reflects quite strongly in \( S_m^m \), provided \( k \leq m - 2 \). This cannot be improved to include the case of \( k = m - 1 \), as the following proposition shows:

1.6 **Proposition.** Let \( m \geq 2 \). Either

(a) for all \( k < m - 1 \) there exists a stationary set \( S \subseteq S_k^m \) that does not reflect fully in \( S_{m-1}^m \),

or

(b) for all \( n > m \) there exists a stationary set \( A \subseteq S_{m-1}^n \) that does not reflect at any \( \delta \in S_m^n \).

We shall give a proof of 1.6 in Section 3. In our model we have, for every \( m \geq 2 \), full reflection of subsets of \( S_0^m \) in \( S_{m-1}^m \) (and of subsets of \( S_k^m \) for \( k \leq m - 3 \)) and therefore 1.6 (a) fails in the model. Thus the model necessarily satisfies 1.6 (b), which shows that the consistency result is best possible.

2. **Proof of Main Theorem**

Let \( \kappa_2 < \kappa_3 < \cdots < \kappa_n < \cdots \) be a sequence of cardinals with the property that for each \( n \geq 2 \), \( \kappa_n \) is a \( < \kappa_{n+1} \)-supercompact cardinal, i.e. for every \( \gamma < \kappa_{n+1} \) there exists an elementary embedding \( j : V \to M \) with critical point \( \kappa_n \) such that \( j(\kappa_n) > \gamma \) and \( M^\gamma \subset M \). We construct the generic extension by iterated forcing, an iteration of length \( \omega \) with full support. The first stage of the iteration \( P_1 \) makes \( \kappa_2 = \aleph_2 \), and for each \( n \), the \( n^{th} \) stage \( P_n \) (a forcing notion in \( V(P_1 \ast \cdots \ast P_{n-1}) \)) makes \( \kappa_{n+1} = \aleph_{n+1} \). In the iteration, we repeatedly use three standard notions of forcing: \( Col(\kappa, \alpha), C(\kappa) \) and \( CU(\kappa, T) \).

**Definition.** Let \( \kappa \) be a regular uncountable cardinal.

(a) \( Col(\kappa, \alpha) \) is the forcing that collapses \( \alpha \geq \kappa \) with conditions of size \( < \kappa \):
A condition is a function \( p \) from a subset of \( \kappa \) of size \( \kappa < \kappa \) into \( \alpha \); a condition \( q \) is stronger than \( p \) if \( q \supseteq p \).

(b) \( C(\kappa) \) is the forcing that adds a Cohen subset of \( \kappa \): A condition is an \( 0 \)-function \( p \) on a subset of \( \kappa \) of size \( \kappa < \kappa \); a condition \( q \) is stronger than \( p \) if \( q \supseteq p \).

(c) \( CU(\kappa, T) \) is the forcing that shoots a club through a stationary set \( T \subseteq \kappa \):
A condition is a closed bounded subset of \( T \); a condition \( q \) is stronger than \( p \) if \( q \) end-extends \( p \).

The first stage \( P_1 \) of the iteration \( P = \langle P_n : n = 1, 2, \cdots \rangle \) is a forcing of size \( \kappa_2 \) that is \( \omega \)-closed, satisfies the \( \kappa_2 \)-chain condition and collapses each cardinal between \( \aleph_1 \) and \( \kappa_2 \) (it is essentially the Levy forcing with countable conditions.) For each \( n \geq 2 \), we construct (in \( V(P|n) \)) the \( n^{th} \) stage \( P_n \) such that

(2.1) (a) \( |P_n| = \kappa_{n+1} \)
(b) \( P_n \) is \( \aleph_{n-2} \) closed
(c) \( P_n \) satisfies the \( \kappa_{n+1} \)-chain condition
(d) \( P_n \) collapses each cardinal between \( \aleph_n (= \kappa_n) \) and \( \kappa_{n+1} \)
(e) \( P_n \) does not add any \( \omega_{n-1} \)-sequences of ordinals

and such that \( P_n \) guarantees the reflection of stationary subsets of \( \aleph_n \) stated in the theorem.

It follows, by induction, that each \( \kappa_n \) becomes \( \aleph_n \): Assuming that \( \kappa_n = \aleph_n \) in \( V(P|n) \), the \( n^{th} \) stage \( P_n \) preserves \( \aleph_n \) by (e), and the rest of the iteration \( \langle P_{n+1}, P_{n+2}, \cdots \rangle \) also preserves \( \aleph_n \) because it is \( \aleph_{n-1} \)-closed by (b); \( P_n \) makes \( \kappa_{n+1} \) the successor of \( \kappa_n \) by (c) and (d).

We first define the forcing \( P_1 \):

\( P_1 \) is an iteration, with countable support, \( \langle Q_\alpha : \alpha < \kappa_2 \rangle \) where for each \( \alpha \),

\[
Q_\alpha = Col(\aleph_1, \aleph_1 + \alpha) \times C(\aleph_1).
\]

It follows easily from well known facts that \( P_1 \) is an \( \omega \)-closed forcing of size \( \kappa_2 \), satisfies the \( \kappa_2 \)-chain condition and makes \( \kappa_2 = \aleph_2 \).

Next we define the forcing \( P_2 \). (It is a modification of Magidor’s forcing from [2], but the added collapsing of cardinals requires a stronger assumption on \( \kappa_2 \) than weak
compactness. The iteration is padded up by the addition of Cohen forcing which will make the main argument of the proof work more smoothly. The definition of $P_2$ is inside the model $V(P_1)$, and so $\kappa_2 = \aleph_2$.

$P_2$ is an iteration, with $\aleph_1$ - support, $\langle Q_\alpha : \alpha < \kappa_3 \rangle$ where for each $\alpha$,

$$Q_\alpha = \text{Col}(\aleph_2, \aleph_2 + \alpha) \times C(\aleph_2) \times CU(T_\alpha)$$

where $T_\alpha$ is, in $V(P_1 * P_2|\alpha)$, some stationary subset of $\omega_2$. We choose the $T_\alpha$'s so that each $T_\alpha$ contains all limit ordinals of cofinality $\omega$. It follows easily that for each $\alpha < \kappa_3, P_2|\alpha \models Q_\alpha$ is $\omega$-closed.

The crucial property of the forcing $P_2$ will be the following:

**Lemma 2.2.** $P_2$ does not add new $\omega_1$ - sequences of ordinals.

One consequence of Lemma 2.2 is that the conditions $(p, q, s) \in Q_\alpha$ can be taken to be sets in $V(P_1)$ (rather than in $V(P_1 * P_2|\alpha)$). Once we have Lemma 2.2, the properties (2-1) (a) - (e) follow easily.

It remains to specify the choice of the $T_\alpha$'s. By a standard argument using the $\kappa_3$ - chain condition, we can enumerate all potential subsets of $\omega_2$ by a sequence $\langle S_\alpha : \alpha < \kappa_3 \rangle$ in such a way that each $S_\alpha$ is already in $V(P_1 * P_2|\alpha)$. At the stage $\alpha$ of the iteration, we let $T_\alpha = \omega_2$, unless $S_\alpha$ is, in $V(P_1 * P_2|\alpha)$, a stationary set of ordinals of cofinality $\omega$. If that is the case, we let

$$T_\alpha = (\text{Tr}(S_\alpha) \cap S^2_1) \cup S^2_0$$

Assuming that Lemma 2.2 holds, we now show that in $V(P_1 * P_2)$, every stationary $S \subseteq S^2_0$ reflects fully in $S^2_1$:

The set $S$ appears as $S_\alpha$ at some stage $\alpha$, and because it is stationary in $V(P_1 * P_2)$, it is stationary in the smaller model $V(P_1 * P_2|\alpha)$. The forcing $Q_\alpha$ creates a closed unbounded set $C$ such that $C \cap S^2_1 \subseteq \text{Tr}(S)$ (note that because $P_2$ does not add $\omega_1$ - sequences, the meaning of $\text{Tr}(S)$ or of $S^2_1$ does not change).

Thus in $V(P_1 * P_2)$ we have full reflection of subsets of $S^2_0$ in $S^2_1$. The later stages of the iteration do not add new subsets of $\omega_2$ and so this full reflection remains true in $V(P)$.

We postpone the proof of Lemma 2.2 until after the definition of the rest of the iteration.
We now define $P_n$ for $n \geq 3$. We work in $V(P_1 \ast \cdots \ast P_{n-1})$. By the induction hypothesis we have $\kappa_n = \aleph_n$.

$P_n$ is an iteration with $\aleph_{n-1}$-support, $\langle Q_\alpha : \alpha < \kappa_{n+1} \rangle$, where for each $\alpha$,

$$Q_\alpha = \text{Col}(\aleph_n, \aleph_n + \alpha) \times C(\aleph_n) \times CU(T_\alpha)$$

where $T_\alpha$ is a $P_n|\alpha$-name for a subset of $\omega_n$. To specify the $T_\alpha$’s, let $\langle S_\alpha : \alpha < \kappa_{n+1} \rangle$ be an enumeration of all potential subsets of $\omega_n$ such that each $S_\alpha$ is a $P_n|\alpha$-name. At the stage $\alpha$, let $T_\alpha = \omega_n$ unless $S_\alpha$ a stationary set of ordinals and $S_\alpha \subseteq S^n_k$ for some $k = 0, \cdots, n-3$, in which case let

$$T_\alpha = (Tr(S_\alpha) \cap S^n_{n-1}) \cup (S^n_0 \cup \cdots \cup S^n_{n-2})$$

$$= \{ \gamma < \omega_n : cf\gamma \leq \omega_{n-2} \text{ or } S_\alpha \cap \gamma \text{ is stationary} \}$$

Due to the selection of the $T_\alpha$’s, $Q_\alpha$ is $\omega_{n-2}$-closed, and so is $P_n$. The crucial property of the forcing is the analog of Lemma 2.2:

**Lemma 2.3.** $P_n$ does not add new $\omega_{n-1}$-sequences of ordinals.

Given this lemma, properties (2.1) (a) - (e) follow easily. The same argument as given above for $P_2$ shows that in $V(P_1 \ast \cdots \ast P_n)$, and therefore in $V(P)$ as well, every stationary subset of $S^n_k$, $k = 0, \cdots, n-3$, reflects fully in $S^n_{n-1}$.

It remains to prove Lemmas 2.2 and 2.3. We prove Lemma 2.3, as 2.2 is an easy modification

**Proof of Lemma 2.3.**

Let $n \geq 3$, and let us give the argument for a specific $n$, say $n = 4$. We want to show that $P_4$ does not add $\omega_3$-sequences of ordinals.

We will work in $V(P_1 \ast P_2)$ (and so consider the forcing $P_3 \ast P_4$). As $P_1 \ast P_2$ has size $\kappa_3, \kappa_4$ is a $< \kappa_5$-supercompact cardinal in $V(P_1 \ast P_2)$, and $\kappa_3 = \aleph_3$. The forcing $P_3$ is an iteration of length $\kappa_4$ that makes $\kappa_4 = \aleph_4$ and is $\aleph_1$-closed; then $P_4$ is an iteration of length $\kappa_5$. By induction on $\alpha < \kappa_5$ we show

(2.4) \hspace{1cm} \text{$P_4|\alpha$ does not add $\omega_3$-sequences of ordinals.}
As $P_3$ has the $\aleph_5$-chain condition, (2.4) is certainly enough for Lemma 2.3. Let $\alpha < \kappa_5$.

Let $j$ be an elementary embedding $j : V \to M$ (as we work in $V(P_1 \ast P_2)$, $V$ means $V(P_1 \ast P_2)$) such that $j(\kappa_4) > \beta$ and $M^\beta \subseteq M$, for some inaccessible cardinal $\beta > \alpha$. Consider the forcing $j(P_3)$ in $M$. It is an iteration of which $P_3$ is an initial segment.

By a standard argument, the elementary embedding $j : V \to M$ can be extended to an elementary embedding $j : V(P_3) \to M(j(P_3))$. We claim that every $\beta$-sequence of ordinals in $V(P_3)$ belongs to $M(j(P_3))$: the name for such a set has size $\leq \beta$ and so it belongs to $M$, and since $P_3 \in M$ and $M(P_3) \subseteq M(j(P_3))$, the claim follows. In particular, $P_4|\alpha \in M(j(P_3))$.

Let $p, \hat{F} \in V(P_3)$ be such that $p \in P_4|\alpha$ and $\hat{F}$ is a $(P_4|\alpha)$-name for an $\omega_3$-sequence of ordinals. We shall find a stronger condition that decides all the values of $\hat{F}$. By the elementarity of $j$, it suffices to prove that

$$(2.5) \exists \bar{p} \leq j(p) \text{ in } j(P_4|\alpha) \text{ that decides } j(\hat{F}).$$

The rest of the proof is devoted to the proof of (2.5).

Let $G$ be a $M$-generic filter on $j(P_3)$.

**Lemma 2.6.** In $M[G]$ there is a generic filter $H$ on $P_4|\alpha$ over $M[G \cap P_3]$ such that $M[G]$ is generic extension of $M[G \cap P_3][H]$ by an $\aleph_1$-closed forcing, and such that $p \in H$.

**Proof.** There is an $\eta < j(\kappa_4)$ such that $P_4|\alpha$ has size $\aleph_3$ in $M_\eta = M[G \cap (j(P_3)|\eta)]$. Since $P_4|\alpha$ is $\aleph_2$-closed, it is isomorphic in $M_\eta$ to the Cohen forcing $C(\aleph_3)$. But $Q_\eta = (j(P_3))(\eta) = Col(\aleph_3, \aleph_3 + \eta) \times C(\aleph_3) \times CU(T_\eta)$, so $G|Q_\eta = G_{Col} \times G_C \times G_{CU}$, and using $G_C$ and the isomorphism between $P_4|\alpha$ and $C(\aleph_3)$ we obtain $H$. Since $j(P_3)$ is $\aleph_1$-closed, the quotient forcing $j(P_3)/(P_3 \times C(a_3))$ is also $\aleph_1$-closed.

**Lemma 2.7.** In $M[G]$ there is a condition $\bar{p} \in j(P_4|\alpha)$ that extends $p$, and extends every member of $j''H$.

Lemma 2.7 will complete the proof of (2.5): since every value of $\hat{F}$ is decided by some condition in $H$, every value of $j(\hat{F})$ is decided by some condition in $j''H$, and therefore by $\bar{p}$.
Proof of Lemma 2.7. Working in $M[G]$, we construct $\overline{p} \in j(P_4|\alpha)$, a sequence $\langle p_\xi : \xi < j(\alpha) \rangle$ of length $j(\alpha)$, by induction. When $\xi$ is not in the range of $j$, we let $p_\xi$ be the trivial condition; that guarantees that the support of $\overline{p}$ has size $|\alpha|$ which is $\aleph_3$ (because $\alpha < j(\kappa_4) = \aleph_4$ in $M[G]$). So let $\xi < \alpha$ be such that $\overline{p}|j(\xi)$ has been defined, and construct $p_{j(\xi)}$.

The condition $p_{j(\xi)}$ has three parts $u, v, s$ where $u \in Col(j(\kappa_4), j(\kappa_4) + j(\xi)), v \in C((\kappa_4))$ and $s \in CU(T_{j(\xi)})$. It is easy to construct the $u$ - part and the $v$ - part: simply take the union of the $u$ - parts and of the $v$ - parts of the the filter $j(H)|j(P_4)(j(\xi)) = j(H|(P_4(\xi)))$. These are functions of size $\aleph_3$ and therefore members of $Col$ and $C$ respectively. For the $s$ - part, take the union of the $s$ - parts of $j(H|(P_4(\xi)))$, which is a closed subset of $\kappa_4$, and add the point $\kappa_4$ to this set. In order that this set be a condition in $CU(T_{j(\xi)})$, we have to verify that $\kappa_4 \in T_{j(\xi)}$.

This is a nontrivial requirement if $S_{j(\xi)}$ is in $M(j(P_3) \ast (j(P_4)|j(\xi)))$ a stationary subset of $j(\kappa_4)$ and is a subset of either $S_0^4$ or of $S_1^4$ (of $S_k^n$ for $n = 4$ and $k \leq n - 3$). Then $\kappa_4$ has to be reflecting point of $S_{j(\xi)}$, i.e. we have to show that $S_{j(\xi)} \cap \kappa_4$ is stationary, in $M(j(P_3) \ast (j(P_4)|j(\xi)))$.

By the assumption and by elementarity of $j$, $S_\xi$ is a stationary subset of $\kappa_4$ in $V(P_3 \ast P_4|\xi)$, and $S_\xi \subseteq S_0^4$ or $S_\xi \subseteq S_1^4$; i.e. consists of ordinals of cofinality $\leq \omega_1$. Since $S_{j(\xi)} \cap \kappa_4 = j(S_\xi) \cap \kappa_4 = S_\xi$, it suffices to show that $S_\xi$ is stationary not only in $V(P_3 \ast P_4|\xi)$ but also in $M(j(P_3) \ast (j(P_4)|j(\xi)))$.

Firstly $M(P_3 \ast P_4|\xi) \subseteq V(P_3 \ast P|\xi)$, and so $S_\xi$ is stationary in $M(P_3 \ast P_4|\xi)$. Secondly, $j(P_4)$ is $\aleph_1$ - closed, and by Lemma 2.6, $M(j(P_3))$ is an $\aleph_1$ - closed forcing extension of $M(P_3 \ast P_4|\xi)$, and so the proof is completed by application of the following lemma (taking $\kappa = \aleph_0$ or $\aleph_1$, $\lambda = \aleph_4$).

Lemma 2.8 Let $\kappa < \lambda$ be regular cardinals and assume that for all $\alpha < \lambda$ and all $\beta < \kappa, \alpha^\beta < \lambda$. Let $Q$ be a $\kappa$ - closed forcing and $S$ a stationary subset of $\lambda$ of ordinals of cofinality $\kappa$. Then $Q \models S$ is stationary.

This lemma is due to Baumgartner and we include the proof for lack of reference.

Proof of Lemma 2.8. Let $q$ be a condition and let $\dot{C}$ be a $Q$ - name for a closed
unbounded subset of \( \lambda \). We shall find \( \vec{q} \leq q \) and \( \gamma \in S \) such that \( \vec{q} \models \gamma \in \dot{C} \). Let \( M \) be a transitive set such that \( M \) is a model of enough set theory, is closed under \( < \kappa \) - sequences and such that \( M \supseteq \lambda, q \in M, Q \in M, \dot{C} \in M \). Let \( \langle N_\gamma : \gamma < \lambda \rangle \) be an elementary chain of submodels of \( M \) such that each \( N_\gamma \) has size \( < \lambda \), contains \( q, Q \) and \( \dot{C}, N_\gamma \cap \lambda \) is an ordinal, and \( N_{\gamma +1} \) contains all \( < \kappa \) - sequences in \( N_\gamma \). Since \( S \) is stationary, there exists a \( \gamma \in S \) such that \( N_\gamma \cap \lambda = \gamma \). As \( cf \gamma = \kappa \), \( N = N_\gamma \) is closed under \( < \kappa \) - sequences.

Let \( \{ \gamma_\xi : \xi < \kappa \} \) be an increasing sequence with limit \( \gamma \). We construct a descending sequence \( \{ q_\xi : \xi < \kappa \} \) of conditions such that \( q_0 = q \), such that for all \( \xi < \kappa \), \( q_\xi \in N \) and for some \( \beta_\xi \in N \) greater than \( \gamma_\xi, q_{\xi +1} \models \beta_\xi \in \dot{C} \). At successor stages, \( q_{\xi +1} \) exists because in \( N \), \( q_\xi \) forces that \( \dot{C} \) is unbounded. At limit stages \( \eta < \kappa \), the \( \eta \) - sequence \( \langle q_\xi : \xi < \eta \rangle \) is in \( N \) and has a lower bound in \( N \) because \( N \models Q \) is \( \kappa \) - closed.

Since \( Q \) is \( \kappa \) - closed, the sequence \( \langle q_\xi : \xi < \kappa \rangle \) has a lower bound \( \vec{q} \), and because of the \( \beta \)'s, \( \vec{q} \) forces that \( \dot{C} \) is unbounded in \( \gamma \). Therefore \( \vec{q} \models \gamma \in \dot{C} \).

3. Negative results.

We shall now present several negative results on the structure of the relation \( S < T \) below \( \aleph_\omega \). With the exception of the proof of Proposition 1.6, we state the results for the particular case of reflection of subsets of \( S_0^3 \) in \( S_1 \), but the results generalize easily to other cardinalities and other cofinalities.

The first result uses a simple calculation (as in Proposition 1.1):

**Proposition 3.1.** For any \( \aleph_3 \) stationary sets \( A_\alpha \subseteq S_3^3, \alpha < \omega_3 \), there exists a stationary set \( S \subseteq S_0^3 \) such that \( S \notin A_\alpha \) for all \( \alpha \).

**Proof.** Let \( A_\alpha, \alpha < \omega_3 \), be stationary subsets of \( S_1^3 \). There exist \( \aleph_4 \) almost disjoint stationary subsets of \( S_0^3 \); let \( S_i, i < \omega_4 \), be such sets. Assuming that each \( S_i \) reflects fully in some \( A_{\alpha(i)} \), we can find \( \aleph_4 \) of them that reflect fully in the same \( A_\alpha \). Take any \( \aleph_2 \) of them and reduce each by a nonstationary set to get \( \aleph_2 \) pairwise disjoint stationary subsets \( \{ T_\xi : \xi < \omega_2 \} \) of \( S_0^3 \), such that each of them reflects fully in \( A_\alpha \). Hence there are clubs \( C_\xi, \xi < \omega_2 \), such that \( Tr(T_\xi) \supseteq A_\alpha \cap C_\xi \) for every \( \xi \). Let \( \gamma \in \bigcup_{\xi \in \omega_2} C_\xi \cap A_\alpha \). Then every \( T_\xi \)
reflects at $\gamma$, and so $\gamma$ has $\aleph_2$ pairwise disjoint stationary subsets $\{T_\xi \cap \gamma : \xi < \omega_2\}$. This is a contradiction because $\gamma$ has a closed unbounded subset of size $cf\gamma = \aleph_1$.  

The next result uses the fact that under GCH there exists a $\diamondsuit$-sequence for $S_0^3$.

**Proposition 3.2.** (GCH) There exists a stationary set $A \subseteq S_1^3$ that is not the trace of any $S \subseteq S_0^3$; precisely: for every $S \subseteq S_0^3$ the set $A \Delta (Tr(S) \cap S_1^3)$ is stationary.

**Proof.** Let $\langle S_\gamma : \gamma \in S_1^3 \rangle$ be a $\diamondsuit$-sequence for $S_1^3$; it has the property that for every set $S \subseteq \omega_3$, the set $D(S) = \{\gamma \in S_1^3 : S \cap \gamma = S_\gamma\}$ is stationary. Let

$$A = \{\gamma \in S_1^3 : S_\gamma \text{ is nonstationary}\}.$$ 

The set $A$ is stationary because $A \supseteq D(\emptyset)$. If $S$ is any stationary subset of $S_0^3$, then for every $\gamma$ in the stationary set $D(S)$, $\gamma \in A$ iff $\gamma \notin Tr(S)$, and so $D(S) \subseteq A \Delta Tr(S)$.

The remaining negative results use the following theorem from [4] which proves the existence of sets with the “square property”.

**Theorem.** (Shelah) Let $1 \leq k \leq n - 2$. The set $S_k^n$ is the union of $\aleph_{n-1}$ stationary sets $A$, each having the following property. There exists a collection $\{C_\gamma : \gamma \in A\}$ (a “square sequence for $A$”) such that for each $\gamma \in A$, $C_\gamma$ is a club subset of $\gamma$ of order type $\omega_k$, consisting of limit ordinals of cofinality $< \omega_k$, and such that for all $\gamma_1, \gamma_2 \in A$ and all $\alpha$, if $\alpha \in C_{\gamma_1} \cap C_{\gamma_2}$ then $C_{\gamma_1} \cap \alpha = C_{\gamma_2} \cap \alpha$.

Square sequences can be used to construct a number of counterexamples. For instance, if $S_n, n < \omega$, are $\aleph_0$ stationary subsets of $S_0^3$, then $Tr(\bigcup_{n=0}^{\infty} S_n) = \bigcup_{n=0}^{\infty} S_n$. Using a square sequence we get:

**Proposition 3.3.** There is a stationary set $A \subseteq S_1^3$ and stationary subsets $S_i, i < \omega_1$, of $S_0^3$ such that $Tr(S_i) \cap A = \emptyset$ for each $i$ but $Tr(\bigcup_{i < \omega_1} S_i) \supseteq A$.

**Proof.** Let $A$ be a stationary subset of $S_1^3$ with a square sequence $\{C_\gamma : \gamma \in A\}$, and let $S = \bigcup_{\gamma \in A} C_\gamma$. Clearly, $S \subseteq S_0^3$ is stationary, and $Tr(S) \supseteq A$. For each $\xi < \omega_1$, let

$$S_\xi = \{\alpha \in S : \text{order type } (C_\gamma \cap \alpha) = \xi\}$$

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(this is independent of the choice of $\gamma \in A$). For every $\gamma \in S$ and every $\xi < \omega_1$, the set $S_{\xi} \cap C_\gamma$ has exactly one element, and so $S_{\xi}$ does not reflect at $\gamma$. It is easy to see that $\kappa_1$ of the sets $S_{\xi}$ are stationary. [The definition of $S_{\xi}$ is a well known trick]

The argument used in the above proof establishes the following:

**Proposition 3.4.** If a stationary set $A \subseteq S^n_m$ has a square sequence and if $k < m$ then there exists a stationary $S \subseteq S^n_k$ that does not reflect at any $\gamma \in A$.

**Proof of Proposition 1.6.** Let $2 \leq m < n$ and let us assume that (b) fails, i.e. that every stationary set $A \subseteq S^n_{m-1}$ reflects at some $\delta$ of cofinality $\kappa_m$. We shall prove that (a) holds. For each $k < m - 1$ we want a stationary set $S \subseteq S^n_k$ that does not reflect fully in $S^n_{m-1}$. Let $k < m - 1$.

Let $A$ be a stationary subset of $S^n_{m-1}$ that have a square sequence $\{C_\gamma : \gamma \in A\}$. The set $A$ reflects at some $\delta$ of cofinality $\omega_m$. Let $C$ be a club subset of $\delta$ of order type $\omega_m$. Using the isomorphism between $C$ and $\omega_m$, the sequence $\{C_\gamma \cap C : \gamma \in A\}$ becomes a square sequence for a stationary subset $B$ of $S^n_{m-1}$. It follows that there is a stationary subset of $S^n_k$ that does not reflect at any $\gamma \in B$.

The last counterexample also uses a square sequence.

**Proposition 3.5.** (GCH) There is a stationary set $A \subseteq S^3_1$ and $\aleph_4$ stationary sets $S_i \subseteq S^3_0$ such that the sets $\{T_\gamma(S_i) \cap A : i < \omega_4\}$ are stationary and pairwise almost disjoint.

**Proof.** Let $A$ be a stationary subset of $S^3_1$ with a square sequence $\langle C_\gamma : \gamma \in A \rangle$, and let $S = \bigcup_{\gamma \in A} C_\gamma$. Let $\{f_i : i < \omega_4\}$ be regressive functions on $S^3_0 \cup S^3_1$ with the property that for any two $f_i, f_j$, the set $(T, H_i)$ is $\langle o, \gamma \rangle = f_j(\alpha)$ is nonstationary (such a family exists by [3]). For each $i$ and each $\gamma \in A$, the function $f_i$ is regressive on $C_\gamma$ and so there is some $\eta = \eta(i, \gamma) < \gamma$ such that $\{\alpha \in C_\gamma : f_i(\alpha) < \eta\}$ is stationary. Let $T_i, \gamma \subseteq \omega_1$ be the stationary set of $\langle T, H_i \rangle$ defined by $H(\xi) = f_i(\xi^\text{th}$ element of $C_\gamma$). For each $i$, the function on $A$ that to each $\gamma$ assigns $(T_i, H_i, \gamma)$ is regressive, and so constant $= (T_i, H_i)$ on a stationary set. By a
counting argument, $(T_i, H_i)$ is the same for all $i$'s; so w.l.o.g. we assume that they are the same $(T, H)$ for all $i$.

Now we let, for each $i$,

\[ A_i = \{ \gamma \in A : (\forall \alpha \in C_\gamma) \text{ if } \xi = o.t.(C_\gamma \cap \alpha) \in T \text{ then } f_i(\alpha) = H(\xi) \} \]

\[ S_i = \{ \alpha \in S : o.t.(C_\gamma \cap \alpha) \in T \text{ and } (\forall \beta \leq \alpha, \beta \in C_\gamma) \text{ if } \xi = o.t.(C_\gamma \cap \beta) \in T \text{ then } f_i(\beta) = H(\xi) \} \]

By the definition of $T$ and $H$, each $A_i$ is a stationary set, and each $S_i$ reflects at every point of $A_i$. We claim that if $\gamma \in A$ and $S_i \cap \gamma$ is stationary then $\gamma \in A_i$. So let $\gamma \in A$ be such that $S_i \cap \gamma$ is stationary. Let $\xi \in T$ and let $\alpha$ be the $\xi^{th}$ element of $C_\gamma$; we need to show that $f_i(\alpha) = H(\xi)$. As $S_i \cap \gamma$ is stationary, there exists a $\beta \in S_i \cap C_\gamma$ greater than $\alpha$. By the definition of $S_i$, $f_i(\alpha) = H(\xi)$. Thus $\gamma \in A_i$, and $A_i = A \cap Tr(S_i)$.

Finally, we show that the sets $A_i$ are pairwise almost disjoint. Let $C$ be a club disjoint from the set $\{ \alpha : f_i(\alpha) = f_j(\alpha) \}$. We claim that the set $C'$ of all limit points of $C$ is disjoint from $A_i \cap A_j$. If $\gamma \in C'$ then $C \cap \gamma$ is a club in $\gamma$, and so is $C \cap C_\gamma$. Since $T$ is stationary in $\omega_1$, there is a $\xi \in T$ such that the $\xi^{th}$ element $\alpha$ of $C_\gamma$ is in $C$, and therefore $f_i(\alpha) \neq f_j(\alpha)$; it follows that $\gamma$ cannot be both in $A_i$ and in $A_j$. \[\square\]


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