

***K-Theory
and
Noncommutative Geometry***

***Lecture 4
Index Cocycles***

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Today's References

Background reading:

D. Quillen, *Algebra cochains and cyclic cohomology*, Publ. Math. I.H.E.S. **68** (1984), 139–174.

E. Getzler and A. Szenes, *On the Chern character of theta-summable Fredholm modules*, J. Functional Anal., **84** (1989), 343–357.

Today's main theorem is (essentially) from our basic reference

A. Connes and H. Moscovici, *The local index formula in noncommutative geometry*, Geom. Funct. Anal. **5** (1995), 174–243.

but the proof is rather different.

Index Pairings

Let H be a $\mathbb{Z}/2$ -graded Hilbert space: $H = H_0 \oplus H_1$.

Let D be a self-adjoint operator on H , of **odd** grading-degree:

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}.$$

Let A be an algebra of **even** grading-degree operators on H , and suppose that

- each $a \in A$ maps $\text{Domain}(D)$ to $\text{Domain}(D)$, and
- the operators $[D, a]: \text{Domain}(D) \rightarrow H$ is *compact*, as are the operators $a(D \pm iI)^{-1}$.

Example. Let (A, H, D) be a spectral triple.

Example. Let $A = C_c^\infty(M)$ and let D be a self-adjoint extension of an order 2 (or higher) elliptic differential operator on M .

If $P \in A$ is an idempotent then $PDP: PH \rightarrow PH$ is an (unbounded) Fredholm operator. Write

$$\text{Index}_\varepsilon(PDP) = \left(\begin{array}{l} \text{Fredholm index of component} \\ \text{of PDP mapping } PH_0 \text{ to } PH_1 \end{array} \right)$$

If A is *unital* we obtain an index map

$$\text{Index}_{\varepsilon, D}: K_0(A) \rightarrow \mathbb{Z}.$$

If A is *nonunital* we obtain a similar map by considering the Fredholm operators

$$\left(\begin{array}{cc} PDP & \varepsilon PQ \\ \varepsilon QP & QDQ \end{array} \right) : PH \oplus QH^{\text{opp}} \rightarrow PH \oplus QH^{\text{opp}},$$

where P and Q are idempotent operators in \tilde{A} whose difference is an operator in A .

Problem. Compute these index maps in various examples arising in noncommutative geometry.

Index Cocycles

Definition. Given A and D , as above, a cyclic cocycle, or (b, B) -cocycle Φ for the algebra A is an *index cocycle* for the pair (A, D) if

$$\langle \Phi, \chi \rangle = \text{Index}_{\varepsilon, D}(\chi),$$

“algebraic index” “analytic index”

for all $\chi \in K_0(A)$.

Remarks.

- On the left hand side is the pairing between cyclic theory and K-theory.
- Index cocycles have an obvious *integrality property*.
- Our objective is to construct index cocycles. This *falls short of* proving *index theorems*, for which we typically need to identify an index cocycle with something computable and concrete.

Connes' Cyclic Chern Character

Definition. Let $p \geq 1$. The *Schatten p -ideal* is

$$\mathcal{L}^p(\mathcal{H}) = \left\{ T: \mathcal{H} \rightarrow \mathcal{H} \mid \sum \mu_n(T)^p < \infty \right\}.$$

One has $\mathcal{L}^1(\mathcal{H}) \subseteq \mathcal{L}^p(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$.

There is a version of *Holder's Inequality*:

$$\mathcal{L}^p(\mathcal{H}) \cdot \mathcal{L}^q(\mathcal{H}) \subseteq \mathcal{L}^r(\mathcal{H}), \quad \text{where } \frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Definition. A spectral triple (A, \mathcal{H}, D) is *p -summable* if $a(I + \Delta)^{-\frac{1}{2}} \in \mathcal{L}^p(\mathcal{H})$, for all $a \in A$.

Lemma. Assume that (A, \mathcal{H}, D) is p -summable, and, for simplicity, that D is invertible. Let

$$F = D\Delta^{-\frac{1}{2}}.$$

Then $[a, F] \in \mathcal{L}^p(\mathcal{H})$, for every $a \in A$.

Remarks. One has

$$F^2 = D\Delta^{-\frac{1}{2}}D\Delta^{-\frac{1}{2}} = D^2\Delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}} = D^2\Delta^{-1} = I.$$

If $Dv = \lambda v$ then $Fv = \text{sign}(\lambda)v$.

Theorem (Connes). *If (A, H, D) is p -summable and if $n \geq p$ then the formula*

$$\phi_n(a^0, \dots, a^n) = (-1)^{\frac{n}{2}} \text{Trace}(\varepsilon a^0 [F, a^1] \dots [F, a^n])$$

defines a cyclic n -cocycle on A .

Remark. If n is odd then ϕ_n is identically zero.

Remark. An adaptation deals with the case where D is *not* invertible.

Theorem (Connes). *If $n \geq p$ is **even** then ϕ_n is an index cocycle for (A, D) . Thus*

$$\text{Index}(PDP: PH_0 \rightarrow PH_1) = (-1)^{\frac{n}{2}} \text{Trace}(\varepsilon P[F, P]^n).$$

Remark. Note the similarity with the formula $\int \text{Trace}(PdP^n)$ for characteristic numbers.

A Small Refinement

Definition. Denote by $\text{Trace}'(X)$ the quantity

$$\text{Trace}'(X) = \frac{1}{2} \text{Trace}(\varepsilon F[F, X])$$

Note. If X is trace-class then $\text{Trace}'(X) = \text{Trace}(X)$.

Theorem. *If (A, H, D) is p -summable and $n \geq p - 1$ then the formula*

$$\begin{aligned} \phi_n(\alpha^0, \dots, \alpha^n) &= (-1)^{\frac{n}{2}} \text{Trace}'(\alpha^0[F, \alpha^1] \dots [F, \alpha^n]) \\ &= (-1)^{\frac{n}{2}} \text{Trace}(\varepsilon F[F, \alpha^0][F, \alpha^1] \dots [F, \alpha^n]) \end{aligned}$$

defines a cyclic n -cocycle on A which is an index cocycle for (A, D) .

Proof when $n = 0$. The cocycle condition is that $\alpha \mapsto \text{Trace}'(\alpha)$ is a trace on A , which is easily checked. The index formula amounts to

$$\text{Trace}(P - Q) = \text{Index}(QP: PH \rightarrow QH)$$

when P, Q are idempotents with $P - Q \in \mathcal{L}^1(H)$. \square

The Chern Character as a (b, B) -Cocycle

According to our conventions the $2k$ -th cyclic Chern character determines the (b, B) -cocycle

$$\Phi_n = \begin{cases} (-1)^{\frac{n}{2}} \frac{\Gamma(\frac{n}{2} + 1)}{n!} \phi_n & \text{if } n = 2k \\ 0 & \text{if } n \neq 2k \end{cases}$$

The following computation helps confirm that our conventions are reasonable.

Lemma. *The cyclic cohomology class of the (b, B) -cocycle Φ is independent of the choice of $k \geq p/2$.*

Proof. Let Ψ_{n+1} be the cochain

$$(-1)^{n+1} \frac{\Gamma(\frac{n}{2} + 2)}{(n+2)!} \text{Trace}(\varepsilon a^0 F[F, a^1][F, a^2] \dots [F, a^{n+1}]).$$

Then one checks that $b\Psi_{n+1} = (-1)^{\frac{n+2}{2}} \frac{\Gamma(\frac{n+2}{2} + 1)}{(n+2)!} \Phi_{n+2}$
 while $B\Psi_{n+1} = -(-1)^{\frac{n}{2}} \frac{\Gamma(\frac{n}{2} + 1)}{n!} \Phi_n. \quad \square$

The JLO cocycle

(J = Jaffe, L = Lesniewski and O = Osterwalder.)

Let (A, H, D) be a spectral triple and assume that

$$\text{Trace}(e^{-t\Delta}) < \infty, \quad \text{for all } t > 0.$$

This is *theta summability*, a very weak condition.

Theorem (JLO). *The formula*

$$\Phi_n(a^0, \dots, a^n) = \int_{\Sigma^n} \text{Trace} (\varepsilon a^0 e^{-t_0 \Delta} [D, a^1] e^{-t_1 \Delta} \dots e^{-t_{n-1} \Delta} [D, a^n] e^{-t_n \Delta}) dt$$

for $n = 0, 2, 4, \dots$ defines an (improper) index cocycle in the (b, B) -bicomplex. (Here Σ^n is the standard n -simplex.)

Note. There are convergence problems to be addressed in pairing the cocycle with K-theory. This is the domain of *entire cyclic cohomology*.

Quillen's Approach to JLO

This is formal (it **ignores analysis**), but at the same time very suggestive. First, some background:

Lemma. *In a Banach algebra,*

$$e^{a+b} = \sum_{n=0}^{\infty} \int_{\Sigma^n} e^{-t_0 a} b e^{-t_1 a} b \dots b e^{-t_n a} dt. \quad \square$$

This follows from Duhamel's equation

$$e^{s(a+b)} = e^{sa} + \int_0^s e^{t(a+b)} b e^{(s-t)a} dt$$

and an iteration argument.

Is the JLO cocycle (a trace of) an exponential?

In the last lecture, we gave $\text{Hom}^{**}(A, L)$ the structure of a (differential) $\mathbb{Z}/2$ -graded algebra:

$$\begin{aligned} \phi \vee \psi(a^1, \dots, a^n) \\ = \sum_{p+q=n} \phi(a^1, \dots, a^p) \psi(a^{p+1}, \dots, a^n) \end{aligned}$$

Denote by $\partial_H(\phi)$ the $\mathbb{Z}/2$ -grading-degree of ϕ (e.g. $\partial_H(\phi)$ is **even** if the **odd-multilinear components** of ϕ are **zero**).

If L is $\mathbb{Z}/2$ -graded, modify the algebra structure to

$$\phi \diamond \psi = (-1)^{\partial_H(\phi)\partial_L(\psi)} \phi \vee \psi$$

and define $\partial(\phi) = \partial_H(\phi) + \partial_L(\phi)$.

Proposition. $\text{Hom}^{**}(A, L)$ is once again a $\mathbb{Z}/2$ -graded algebra. \square

Remark. The operator $d\phi = (-1)^{\partial_L(\phi)} b' \phi$ is a graded derivation.

Back to spectral triples and the JLO cocycle ...

Let L be the $\mathbb{Z}/2$ -graded algebra of ‘operators’ on $H = H_0 \oplus H_1$. Let $\rho: A \rightarrow L$ be the given representation of A on H .

Define the ‘superconnection form’

$$\theta = D - \rho \in \text{Hom}^{**}(A, L)$$

(of odd-grading-degree) and let K be its ‘curvature’:

$$K = d\theta + \theta^2.$$

Lemma. $K = \Delta - E$, where $E: A \rightarrow L$ is defined by

$$E(a) = [D, \rho(a)]. \quad \square$$

Definition. Denote by $e^{-K} \in \text{Hom}^{**}(A, L)$ the element

$$e^{-K} = \sum_{n=0}^{\infty} \underbrace{\int_{\Sigma^n} e^{-t_0 \Delta} E e^{-t_1 \Delta} \dots E e^{-t_n \Delta} dt}_{\text{n-linear map}}$$

$$a_1, \dots, a_n \mapsto \int_{\Sigma^n} e^{-t_0 \Delta} [D, a^1] e^{-t_1 \Delta} \dots [D, a^n] e^{-t_n \Delta} dt$$

Lemma (Bianchi Identity).

$$d(e^{-K}) + [e^{-K}, \theta] = 0. \quad \square$$

Lemma (Differential Equation). *Suppose that ∂ is a derivation of $\text{Hom}^{**}(A, L)$ into a bimodule. Then*

$$\partial(e^{-K}) = -\partial(K)e^{-K},$$

modulo (limits of) commutators. □

These follow from the ‘Duhamel formula’

$$D(e^{-K}) = \int_0^1 e^{-tK} D(K) e^{-(1-t)K} dt$$

where D is any derivation.

Note. We are disregarding analytic details, for now.

Construction of the JLO cocycle

From a (theta summable) spectral triple (A, H, D) we have constructed a family of multilinear maps

$$(a^1, \dots, a^n) \mapsto \int_{\Sigma^n} e^{-t_0 \Delta} [D, a^1] e^{-t_1 \Delta} [D, a^2] \dots [D, a^n] e^{-t_n \Delta} dt$$

with values in L (actually the trace-class operators).

Suppose now we compose with the ‘supertrace’

$$\text{Trace}_\varepsilon(X) = \text{Trace}(\varepsilon X),$$

Theorem (Quillen). *As a result of the **Bianchi identity** and the **differential equation** satisfied by e^{-K} , the formula*

$$\Phi_{2n}(a^0, \dots, a^{2n}) = \int_{\Sigma^n} \text{Trace} (\varepsilon a^0 e^{-t_0 \Delta} [D, a^1] e^{-t_1 \Delta} [D, a^2] \dots [D, a^n] e^{-t_n \Delta}) dt$$

defines a (b, B) -cocycle. □

The (b, B) -Bicomplex

$$\begin{array}{ccccccc}
 \vdots & \uparrow & & \vdots & \uparrow & & \vdots \\
 b & \uparrow & & b & \uparrow & & b \\
 \text{Hom}(A \otimes A \otimes A, \mathbb{C}) & \xrightarrow{B} & \text{Hom}(A \otimes A \otimes A, \mathbb{C}) & \xrightarrow{B} & \text{Hom}(A \otimes A, \mathbb{C}) & \xrightarrow{B} & \text{Hom}(A, \mathbb{C}) \\
 \vdots & \uparrow & & \vdots & \uparrow & & \vdots \\
 b & \uparrow & & b & \uparrow & & b \\
 \text{Hom}(A \otimes A \otimes A, \mathbb{C}) & \xrightarrow{B} & \text{Hom}(A \otimes A, \mathbb{C}) & \xrightarrow{B} & \text{Hom}(A, \mathbb{C}) & & \\
 \vdots & \uparrow & & \vdots & \uparrow & & \vdots \\
 b & \uparrow & & b & \uparrow & & b \\
 \text{Hom}(A \otimes A, \mathbb{C}) & \xrightarrow{B} & \text{Hom}(A, \mathbb{C}) & & & & \\
 \vdots & \uparrow & & \vdots & \uparrow & & \vdots \\
 b & \uparrow & & b & \uparrow & & b \\
 \text{Hom}(A, \mathbb{C}) & & & & & &
 \end{array}$$

The Connes-Tsygan Complex

$$\begin{array}{ccccccc}
 \vdots & \uparrow & & \uparrow & \vdots & \uparrow & \vdots \\
 b & \uparrow & & b' & b & \uparrow & b \\
 \text{Hom}(A \otimes A \otimes A, \mathbb{C}) & \xrightarrow{N'} & \text{Hom}(A \otimes A \otimes A, \mathbb{C}) & \xrightarrow{N} & \text{Hom}(A \otimes A \otimes A, \mathbb{C}) & \xrightarrow{N'} & \dots \\
 b & \uparrow & & b' & b & \uparrow & b \\
 \text{Hom}(A \otimes A, \mathbb{C}) & \xrightarrow{N'} & \text{Hom}(A \otimes A, \mathbb{C}) & \xrightarrow{N} & \text{Hom}(A \otimes A, \mathbb{C}) & \xrightarrow{N'} & \dots \\
 b & \uparrow & & b' & b & \uparrow & b \\
 \text{Hom}(A, \mathbb{C}) & \xrightarrow{N'} & \text{Hom}(A, \mathbb{C}) & \xrightarrow{N} & \text{Hom}(A, \mathbb{C}) & \xrightarrow{N'} & \dots
 \end{array}$$

$$b' \phi(a^0, \dots, a^{n+1}) = \sum_{i=0}^n \phi(a^0, \dots, a^i a^{i+1}, \dots, a^{n+1})$$

$$N = 1 + \lambda + \dots + \lambda^n \quad \text{and} \quad N' = 1 - \lambda.$$

$$bN = Nb' \quad \text{and} \quad b'N' = N'b.$$

Remark. The two complexes are essentially the same. If we define t by

$$t\phi(a^0, \dots, a^n) = (-1)^n \phi(a^0, \dots, a^n, 1)$$

then $B = NtN'$.

Quillen uses the Connes-Tsygan complex.

Definition. $\Phi \in \text{Hom}^{**}(A, \mathbb{C})$ and $\phi \in \text{Hom}^{**}(A, \mathbb{C})$ comprise a *cyclic pair* if

$$\clubsuit \quad b\Phi + N\phi = 0$$

$$\heartsuit \quad b'\phi + N'\Phi = 0$$

Definition. Suppose that the algebra A is unital. A pair of elements $\Phi \in \text{Hom}^{**}(A, \mathbb{C})$ and $\phi \in \text{Hom}^{**}(A, \mathbb{C})$ is *normalized* if

$$\phi_n(a^1, \dots, a^n) = \Phi(1, a^1, \dots, a^n)$$

$$\Phi_n(a^0, \dots, a^n) = 0 \quad \text{whenever } a^i = 1 \text{ for some } i \geq 1.$$

Lemma. *If (Φ, ϕ) is a normalized cyclic pair then the families*

$$(\Phi_0, \Phi_2, \Phi_4, \dots) \quad \text{and} \quad (\Phi_1, \Phi_3, \Phi_5, \dots)$$

*are respectively even and odd **improper** cocycles in the periodic (b, B) -complex. \square*

By **improper** we mean that the families are not necessarily eventually zero (as periodic cocycles should be).

Quillen shows that from e^{-K} is obtained a cyclic pair. (The proof uses a remarkable identification of the odd rows in the Connes-Tsygan complex as commutator quotients.)

♡ corresponds to the Bianchi identity.

♣ corresponds to the differential equation.

The corresponding (b, B) -cocycle is the JLO cocycle.

Residue Cocycle – Preliminaries

We shall work with an *admissible* pseudodifferential operator algebra $\Psi^a(\mathcal{D}, \Delta) \triangleleft \Psi(\mathcal{D}, \Delta)$, as follows:

- H is $\mathbb{Z}/2$ -graded, $H = H_0 \oplus H_1$, and the grading operator $\varepsilon = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ belongs to \mathcal{D}_0 .
- $\Delta = D^2$, where $D = \begin{pmatrix} 0 & D_+^* \\ D_+ & 0 \end{pmatrix}$, and $D \in \mathcal{D}_r$, $r = \frac{p}{2}$.
- $\mathcal{A} \subseteq \mathcal{D}_0$ is a subalgebra of grading-degree zero operators, with $[\mathcal{A}, D] \subseteq \mathcal{D}_{r-1}$.
- If $T \in \Psi_0^a(\mathcal{D}, \Delta)$ then $T(I + \Delta)^{-s}$ is trace-class for all $s > \frac{d}{p}$.
- The zeta functions $\zeta(s) = \text{Trace}(T(I + \Delta)^{-s})$ extend to meromorphic functions on \mathbb{C} , with only simple poles. Denote by $\tau: \Psi^a(\mathcal{D}, \Delta) \rightarrow \mathbb{C}$ the residue trace.

Other variations are possible.

Connes-Moscovici Index Theorem

Theorem. Let $\Psi^a(\mathcal{D}, \Delta)$ be an admissible pseudodifferential operator algebra. The formula

$$\Phi_n(a^0, \dots, a^n) = \sum_{k \geq 0} c_{nk} \tau(\varepsilon a_0 [D, a^1]^{(k_1)} [D, a^2]^{(k_2)} \dots [D, a^n]^{(k_n)} \Delta^{-\frac{n+2k}{2}})$$

where $n = 0, 2, \dots$ and

$$c_{nk} = \frac{(-1)^{|k|} \Gamma(|k| + \frac{n}{2})}{k! (k_1 + 1)(k_1 + k_2 + 2) \dots (k_1 + \dots + k_n + n)}.$$

defines an index cocycle for the pair (D, A) in the (b, B) -bicomplex.

Remarks

- $k = (k_1, \dots, k_n)$ is a multi-index with nonnegative integer components.
- $\tau(\mathbb{T}\Delta^{-\frac{n+2k}{2}})$ is defined since $\Delta^{\frac{1}{2}}$ is invertible in $\Psi(\mathcal{D}, \Delta)$, modulo very low order operators.
- Note that if $n + k > d$ then the (n, k) -contribution to the index formula is **identically zero**. Thus the sum in the formula is finite, for each n , and is identically zero, for $n > d$.
- c_{00} is not well defined by the above formula since the Γ -function has a pole at $z = 0$. If Δ is invertible we let

$$c_{00}\tau(\varepsilon a^0) = \frac{1}{2} \operatorname{Res}_{s=0} \left(\Gamma(s) \operatorname{Trace}(\varepsilon a^0 \Delta^{-s}) \right).$$

- If Δ is not invertible this definition must be altered.

Classical Case

Let M^d be a (complete) even-dimensional Spin manifold, let D be the Dirac operator, and form the pseudodifferential operator algebra $\Psi(M, \Delta)$. Denote by $\Psi_n^a(M, \Delta)$ the operators T which for every k may be written

$$T = D\Delta^{-\frac{n+m}{2}} + R,$$

where D is a compactly supported, order m differential operator and R is trace-class as an operator from H^s to H^{s+k} , for all s (in particular, $\text{order}_\Delta(R) < -k$).

Theorem. *In the classical case, the $k \neq 0$ terms in the Connes-Moscovici formula vanish. Moreover*

$$\begin{aligned} \tau(\varepsilon a_0 [D, a^1] [D, a^2] \dots [D, a^n] \Delta^{-\frac{n}{2}}) \\ = \text{constant} \cdot \int_M a^0 da^1 \dots da^n \wedge \widehat{A}(M). \end{aligned}$$

This follows from Getzler's approach to the Atiyah Singer Theorem.

Typical Case (Complexity Estimate)

In the *simplest* case of interest to Connes and Moscovici one has

- $\Gamma \subseteq \text{Diffeo}^+(\mathbb{R})$
- $A = C_c^\infty(\mathbb{R}^2) \rtimes \Gamma$ (the crossed product algebra),
 $g(x, t) = (g(x), t + \log(g'(x)))$.
- $D = \begin{pmatrix} e^t \frac{\partial}{\partial x} & \frac{\partial^2}{\partial t^2} \\ \frac{\partial^2}{\partial t^2} & -e^t \frac{\partial}{\partial x} \end{pmatrix}$ (roughly speaking).

A typical generator of A looks like $f \cdot g$, and

$$[D, f \cdot g] = \underbrace{f[D, g]}_{2 \text{ terms}} + \underbrace{[D, f]g}_{3 \text{ terms}}$$

The terms are of the form $f \cdot g$, or worse, and (by my rough count)

$$[\Delta, f \cdot g] = \underbrace{f[\Delta, g]}_{4 \text{ terms}} + \underbrace{[\Delta, f]g}_{9 \text{ terms}}$$

Thus $[D, f \cdot g]^{(1)}$ has say 65 terms. **The full CM formula has $\gg 500$ terms!**

Residue Cocycle: Conceptual Approach

We shall emulate Quillen's approach to JLO: we shall construct other functions of K — the complex powers.

Lemma. *If $\lambda \notin \text{Spec}(\Delta)$ then $\lambda - K$ is invertible in $\text{Hom}^{**}(\Lambda, L)$.*

Proof. Since $(\lambda - K) = (\lambda - \Delta + E)$ we can write

$$\begin{aligned}
 (\lambda - K)^{-1} &= (\lambda - \Delta)^{-1} \\
 &\quad \text{0-linear} \\
 &\quad - (\lambda - \Delta)^{-1} E (\lambda - \Delta)^{-1} \\
 &\quad \quad \text{1-linear} \\
 &\quad \quad + (\lambda - \Delta)^{-1} E (\lambda - \Delta)^{-1} E (\lambda - \Delta)^{-1} \\
 &\quad \quad \quad \text{2-linear} \\
 &\quad \quad \quad - \dots
 \end{aligned}$$

The infinite series has an obvious meaning in $\text{Hom}^{**}(\Lambda, L)$. □

For simplicity let us now assume Δ is invertible.

Definition. For any complex s with positive real part define $K^{-s} \in \text{Hom}^{**}(\Lambda, L)$ by

$$K^{-s} = \frac{1}{2\pi i} \int_{\ell} \lambda^{-s} (\lambda - K)^{-1} d\lambda,$$

where ℓ is a vertical line between 0 and $\text{Spec}(\Delta)$. Thus the degree n component of K^{-s} is

$$(a^1, \dots, a^n) \mapsto \frac{(-1)^n}{2\pi i} \int_{\ell} \lambda^{-s} (\lambda - \Delta)^{-1} [D, a^1] \dots [D, a^n] (\lambda - \Delta)^{-1} d\lambda$$

The assumption that $\text{Re}(s) > 0$ guarantees convergence of the integral.

If D is a derivation then

$$D(K^{-s}) = \frac{1}{2\pi i} \int_{\ell} \lambda^{-s} (\lambda - K)^{-1} D(K) (\lambda - K)^{-1} d\lambda.$$

Therefore ...

Lemma (Bianchi Identity). $d(K^{-s}) + [K^{-s}, \theta] = 0$. □

Lemma (Differential Equation). *Suppose that ∂ is a derivation of $\text{Hom}^{**}(A, L)$ into a bimodule. Then*

$$\partial(K^{-s}) = -s\partial(K)K^{-s-1},$$

modulo (limits of) commutators. □

It is convenient to work with $\Gamma(s)K^{-s}$, so that the differential equation becomes

$$\partial(\Gamma(s)K^{-s}) = -\partial(K)\Gamma(s+1)K^{-(s+1)}$$

modulo (limits of commutators). Except for the appearance of $s+1$ in place of s this is the **same as the differential equation for e^{-K}** . Meanwhile the **Bianchi identity**

$$d(\Gamma(s)K^{-s}) + [\Gamma(s)K^{-s}, \theta] = 0$$

still holds.

Following Quillen's approach to JLO one obtains:

Theorem. For $\text{Re}(s) > \frac{n}{2}$ define

$$\phi_n^s(a^0, \dots, a^n) = \frac{(-1)^n \Gamma(s - \frac{n}{2})}{2\pi i} \text{Trace} \left(\int_{\ell} \lambda^{\frac{n}{2}-s} \varepsilon a^0 (\lambda - \Delta)^{-1} [D, a^1] \dots [D, a^n] (\lambda - \Delta)^{-1} d\lambda \right)$$

Then $b\phi_n^s + B\phi_{n+2}^s = 0$.

We have *discovered* this theorem using Quillen's formalism.¹ But having discovered it it is not hard at all to *prove* the theorem *directly*. Getzler and Szenes follow this approach for JLO, and their arguments may be copied here.

¹Actually it is easier here to make Quillen's approach rigorous than it is for JLO.

Residues

We do **not** yet have an cohomologically interesting cocycle since ϕ^s is not finitely supported (or even more than ‘locally’ defined, thanks to the restriction $\operatorname{Re}(s) > \frac{n}{2}$). However:

Proposition. *The function $s \mapsto \phi_n^s(a^0, \dots, a^n)$ is **meromorphic**. Moreover if $n > d$ then this function is **holomorphic near the origin**. \square*

We shall see why this is so in a moment.

Proposition. *The formula*

$$\Phi_n(a^0, \dots, a^n) = \operatorname{Res}_{s=0} \left(\phi_n^s(a^1, \dots, a^n) \right)$$

for $n = 0, 2, 4, \dots$ defines a properly supported (b, B) -cocycle. \square

This is clear: $\operatorname{Res}_{s=0}$ is a linear functional on meromorphic functions, so the cocycle condition is preserved. Moreover there are no poles when $n > d$ so $\Phi_n = 0$ here.

Proposition. *The cocycle Φ may be evaluated as follows:*

$$\Phi(a^0, \dots, a^{2n}) = \sum_k c_{nk} \tau(\varepsilon[D, a^1]^{(k_1)} \dots [D, a^n]^{(k_n)} \Delta^{-\frac{n}{2}-k}),$$

where the constants are as in the CM index theorem.

Note. It is not at all easy to prove **directly** that the formula for Φ above is a (b, B) -cocycle.

Proof. The idea is to move all the terms $(\lambda - \Delta)^{-1}$ in the integral defining ϕ^s to the right, using the binomial formula, then take residues. Thus we start from

$$\begin{aligned} (\lambda - \Delta)^{-h} E &= E(\lambda - \Delta)^{-h} + h \cdot E^{(1)}(\lambda - \Delta)^{-(h+1)} \\ &\quad + \frac{h(h+1)}{2!} E^{(2)}(\lambda - \Delta)^{-(h+2)} + \dots \\ &= \sum_{k \geq 0} \frac{h(h+1) \dots (h+k-1)}{k!} E^{(k)}(\lambda - \Delta)^{-(h+k)}. \end{aligned}$$

Setting $h = 1$ we get

$$(\lambda - \Delta)^{-1} E_1 = \sum_{k_1 \geq 0} C(k_1) E_1^{(k_1)} (\lambda - \Delta)^{-(k_1+1)}$$

where $C(k_1) = \frac{k_1!}{k_1!} = 1$. Next, we get

$$(\lambda - \Delta)^{-1} E_1 (\lambda - \Delta)^{-1} E_2 = \sum_{k_1, k_2 \geq 0} C(k_1, k_2) E_1^{(k_1)} E_2^{(k_2)} (\lambda - \Delta)^{-(|k|+2)}$$

where

$$\begin{aligned} C(k) &= \frac{k_1! (k_1 + 2)(k_1 + 3) \dots (k_1 + k_2 + 1)}{k_1! k_2!} \\ &= \frac{(k_1 + k_2 + 2)!}{k_1! k_2! (k_1 + 1)(k_1 + k_2 + 2)}. \end{aligned}$$

Finally

$$\begin{aligned} (\lambda - \Delta)^{-1} E_1 \dots (\lambda - \Delta)^{-1} E_n \\ = \sum_{k \geq 0} C(k_1, \dots, k_n) E_1^{(k_1)} \dots E_n^{(k_n)} (\lambda - \Delta)^{-(|k|+n)} \end{aligned}$$

where

$$C(k_1, \dots, k_n) = \frac{(k_1 + \dots + k_n + n)!}{k_1! \dots k_n! (k_1 + 1) \dots (k_1 + \dots + k_n + n)}.$$

It follows that

$$\begin{aligned} & \int \lambda^{\frac{n}{2}-s} a^0 (\lambda - \Delta)^{-1} E_1 \dots (\lambda - \Delta)^{-1} E_n (\lambda - \Delta)^{-1} d\lambda \\ &= \sum_{k \geq 0} C(k) E_1^{(k_1)} \dots E_n^{(k_n)} \int \lambda^{\frac{n}{2}-s} (\lambda - \Delta)^{-(|k|+n+1)} d\lambda \end{aligned}$$

Now we can use Cauchy's integral formula to compute

$$\begin{aligned} & \frac{1}{2\pi i} \int \lambda^{\frac{n}{2}-s} (\lambda - \Delta)^{-(|k|+n+1)} d\lambda \\ &= \frac{(\frac{n}{2} - s)(\frac{n}{2} - s - 1) \dots (\frac{n}{2} - s - n - |k| + 1)}{(|k| + n)!} \Delta^{-\frac{n}{2}-|k|-s} \end{aligned}$$

It therefore follows from the functional equation for $\Gamma(s)$ that

$$\begin{aligned} & \frac{(-1)^n \Gamma(s - \frac{n}{2})}{2\pi i} \int \lambda^{\frac{n}{2}-s} (\lambda - \Delta)^{-(|k|+n+1)} d\lambda \\ &= (-1)^n \Gamma(s - \frac{n}{2}) \frac{(\frac{n}{2} - s)(\frac{n}{2} - s - 1) \dots (\frac{n}{2} - s - n - |k| + 1)}{(|k| + n)!} \Delta^{-\frac{n}{2}-|k|-s} \\ &= (-1)^{|k|} \Gamma(s + \frac{n}{2} + |k|) \frac{1}{(|k| + n)!} \Delta^{-\frac{n}{2}-|k|-s} \end{aligned}$$

Putting everything together we get

$$\begin{aligned} & \frac{(-1)^n}{2\pi i} \Gamma\left(\frac{n}{2}-s\right) \text{Trace} \left(\int_{\ell} \lambda^{\frac{n}{2}-s} a^0 (\lambda - \Delta)^{-1} E_1 \dots E_n (\lambda - \Delta)^{-1} d\lambda \right) \\ &= \sum_{k \geq 0} (-1)^{|k|} \Gamma\left(s + \frac{n}{2} + |k|\right) \frac{1}{(|k| + n)!} c(k) \\ & \quad \times \text{Trace} \left(\varepsilon a^0 E_1^{(k_1)} \dots E_n^{(k_n)} \Delta^{-\frac{n}{2}-|k|-s} \right) \end{aligned}$$

and taking residues at $s = 0$ we get the result. \square

Remark. The formula for ϕ^s obtained toward the end of the proof shows that $\phi^s(a^0, \dots, a^n)$ is meromorphic since in any half plane it is a finite sum of zeta functions (each assumed to be meromorphic), plus a holomorphic ‘error’ term.

Homotopy Invariance

Proposition. *If D_t is a smooth homotopy of order one operators, all equal modulo order zero operators, then the residue cocycles associated to the operators D_t are all cohomologous.*

Idea of the Proof. We first find $\psi^{s,t}$ so that

$$b\psi^{s,t} + B\psi^{s,t} = \frac{d}{dt}\phi^{s,t}.$$

We obtain it from the integral

$$\frac{\Gamma(s)}{2\pi i} \int_{\ell} \lambda^{-s} (\lambda - K_t)^{-1} \frac{dD_t}{dt} (\lambda - K_t)^{-1} d\lambda$$

in $\text{Hom}^{**}(A, L)$. Then we integrate $\psi^{s,t}$ from $t = 0$ to $t = 1$ to solve

$$b\psi^s + B\psi^s = \phi^{s,1} - \phi^{s,0}.$$

Then we take residues at $s = 0$. □

Index Formula

Theorem. *The residue cocycle Φ of Connes and Moscovici is an index cocycle.*

Proof. Given $P \in \mathcal{A}$, we want to evaluate the pairing $\langle \Phi, P \rangle$ and obtain $\text{Index}_{\varepsilon, D}(P)$.

Replace D by $PDP + P^\perp D P^\perp$.

The straight line from D to its replacement is a homotopy to which the previous proposition applies. So we can assume that P and D commute. Now the result is easy. \square

Remark. The argument requires that we deal with operators which are not invertible (we have not discussed this nuance here).

Remark. The same argument proves that the JLO cocycle is an index cocycle. This is the approach of Getzler and Szenes.