K-Theory and Noncommutative Geometry

Lecture 3
Cyclic Cohomology

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Preview

Proposition. *If* $\tau\colon A \to \mathbb{C}$ *is a trace on an algebra* A *then the formula*

$$\tau_*[P] = \sum_i \tau(P_{ii}),$$

where P *is an idempotent matrix over* A*, determines a homomorphism*

$$\tau_*\colon K_0(A) \to \mathbb{C}.$$

Proof. Suppose $UV = P$ and $VU = Q$. Then

$$\begin{aligned}
\tau_*[P] &= \sum_i \tau(P_{ii}) \\
&= \sum_{i_1,i_2} \tau(U_{i_1i_2}V_{i_2i_1}) \\
&= \sum_{i_1,i_2} \tau(V_{i_2i_1}U_{i_1i_2}) \\
&= \sum_i \tau(Q_{ii}) = \tau_*[Q].
\end{aligned}$$

□

Proposition (Connes). *If ϕ is a 3-linear functional on A and if*

(a) $\phi(a^0, a^1, a^2) = \phi(a^2, a^0, a^1)$, *and*

(b)
$$\phi(a^0a^1, a^2, a^3) - \phi(a^0, a^1a^2, a^3) + \phi(a^0, a^1, a^2a^3) - \phi(a^3a^0, a^1, a^2) = 0,$$

then the formula

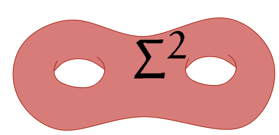
$$\phi_*[P] = \sum_{i_1,i_2,i_3} \phi(P_{i_1i_2}, P_{i_2i_3}, P_{i_3i_1})$$

determines a homomorphism

$$\phi_*\colon K_0(A) \to \mathbb{C}.$$

Proof. Exercise! □

Example. $A = C^\infty(\Sigma)$ and



$$\phi(f^0, f^1, f^2) = \int_\Sigma f^0 df^1 df^2.$$

Characteristic Numbers

Let M be a smooth manifold, $V \subseteq M$ an oriented, closed submanifold. Let $P \colon M \to M_K(\mathbb{C})$ be a smooth, projection-valued function. Define

$$c_V(P) = \int_V \mathrm{Trace}(PdPdP \cdots dPdP).$$

Proposition. *Fixing* V, *the scalar* $c_V(P)$ *only depends on the class* $[P] \in K^0(M)$.

Proof. First, $\mathrm{Trace}(PdPdP \cdots dPdP)$ is a closed form. Second, given a projection-valued function $P \colon I \times M \to M_n(\mathbb{C})$, one has (by Stokes' Theorem)

$$\int_{\partial I \times M} \mathrm{Trace}(PdPdP \cdots dPdP)$$
$$= \int_{I \times M} \mathrm{Trace}(dPdPdP \cdots dPdP) = 0.$$

□

Remark. If $\dim(V)$ is odd then $c_V(P) = 0$ (in fact the differential form $\mathrm{Trace}(PdPdP \cdots dP) \equiv 0$).

Noncommutative Generalization

A = any algebra, and $P \in M_K(A)$, $P^2 = P$.

Question. If $c_V(P) = \int_V \mathrm{Trace}(PdPdP \cdots dPdP)$ then ... *What is V? What is $\int$? What is d?*

Definition (Connes). An n-*cycle* over an algebra A is a pair $(\Omega, \int)$, where

(a) Ω is a differential graded algebra, equipped with an algebra map from A into Ω^0, and

(b) $\int : \Omega^n \to \mathbb{C}$ is a *closed, graded trace* on Ω^*:

(i) $\int \omega_1\omega_2 = (-1)^{\deg(\omega_1)\deg(\omega_2)} \int \omega_2\omega_1$,
(ii) $\int d\omega = 0$.

Remark. It is not necesarily true that $d1 = 0$, nor that $1 \cdot \omega = \omega$, nor that $\omega_1\omega_2 = \pm\omega_2\omega_1$.

Proposition. *If $(\Omega, \int)$ is an n-cycle then the characteristic number*

$$c(P) = \int \mathrm{Trace}(PdPdP \cdots dPdP)$$

depends only on $[P] \in K_0(A)$. □

Cyclic Cocycles

Proposition. *Let* $(\Omega, \int)$ *be an* n*-cycle for* A*. The formula*

$$\varphi(a^0, a^1, \ldots, a^n) = \int a^0 da^1 \cdots da^n$$

defines a multlinear functional on A *with the following properties:*

- $\varphi(a^0, a^1, \ldots, a^n) = (-1)^n \varphi(a^1, \ldots, a^n, a^0)$
- $b\varphi(a^0, \ldots, a^{n+1}) = 0$*, where*

$$\begin{aligned} b\varphi(a^0, \ldots, a^{n+1}) = {} & \varphi(a^0 a^1, \ldots, a^{n+1}) \\ & - \varphi(a^0, a^1 a^2, \ldots, a^{n+1}) \\ & + \ldots \\ & + (-1)^{n+1} \varphi(a^{n+1} a^0, \ldots, a^n). \end{aligned}$$

Definition. Let A be an algebra. A *cyclic* n*-cocycle* on A is an $(n+1)$-multilinear functional on A with the above two properties (cyclicity, coboundary zero).

Proof of the Proposition. Cyclicity is proved as follows. First,

$$\phi(a^0, \dots, a^n) = \int a^0 da^1 \dots da^n$$
$$= \int da^1 \dots da^n \cdot a^0.$$

Next

$$da^n \cdot a^0 = -a^n da^0 + d(a^n a^0),$$

and so (elaborating on this observation)

$$da^1 \dots da^n \cdot a^0 = (-1)^n a^1 da^2 \dots da^0 + \text{exact form}$$

Finally, to prove $b\phi = 0$ use Leibniz's rule:

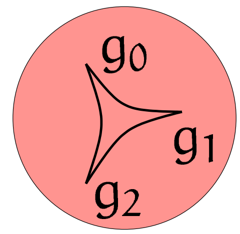
$$\begin{aligned}
&(a^0a^1)da^2 \dots da^{n+1} \\
&\quad -a^0 d(a^1a^2)da^3 \dots da^{n+1} \\
&\qquad + \cdots \\
&\qquad\quad +(-1)^{n+1}(a^{n+1}a^0)da^1 \dots da^n \\
&= (-1)^n \left(a^{n+1}a^0 da^1 \dots da^n - a^0 da^1 \dots da^n \cdot a^{n+1}\right)
\end{aligned}$$

□

Examples of Cyclic Cocycles

Example. Let G be a group and let $c\colon G^{n+1} \to \mathbb{C}$ be a group cocycle. Thus:

$$\begin{cases} c(gg_0, \dots gg_n) = c(g_0, \dots, g_n) \\ \sum (-1)^j c(g_0, \dots, \widehat{g_j}, \dots, g_{n+1}) = 0 \end{cases}$$



The following is a cyclic n-cocycle for $\mathbb{C}[G]$:

$$\varphi_c(g_0, \dots, g_n) = \begin{cases} c(1, g_1, g_1 g_2, \dots) & \text{if } g_0 \cdots g_n = 1 \\ 0 & \text{if } g_0 \cdots g_n \neq 1 \end{cases}$$

Example. Suppose a Lie algebra $\mathfrak{g}$ acts on A by derivations and that τ is an invariant trace on A: $\tau(X(a)) = 0$, $\forall X \in \mathfrak{g}$. Let $c \in \wedge^n \mathfrak{g} \subseteq \otimes^n \mathfrak{g}$ be a (Chevalley-Eilenberg) Lie algebra cycle. If we define

$$\phi_{X^1 \wedge \dots \wedge X^n}(a^0, \dots, a^n) = \tau(a^0 X^1(a^1) \dots X^n(a^n))$$

then ϕ_c is a cyclic n-cocycle on A.

Remark. By 'cycle' we mean a cycle in the Chevalley-Eilenberg complex which computes the homology of $\mathfrak{g}$ with *trivial* coefficients $\mathbb{C}$. The boundary operator in the complex is

$$b(X^1 \wedge \cdots \wedge X^n)$$
$$= \sum_{i<j} (-1)^{i+j} [X^i, X^j] \wedge X^1 \wedge \cdots \wedge \widehat{X^i} \wedge \ldots \widehat{X^j} \wedge \cdots \wedge X^n.$$

We embed the exterior powers $\wedge^n \mathfrak{g}$ into $\otimes^n \mathfrak{g}$ by total antisymmetrization.

Example. If δ^1 and δ^2 are commuting derivations on an algebra A, and if τ is an invariant trace, then the formula

$$\phi(a^0, a^1, a^2) = \tau\left(a^0\left(\delta^1(a^1)\delta^2(a^2) - \delta^2(a^1)\delta^1(a^1)\right)\right)$$

is a cyclic 2-cocycle.

Cyclic Cohomology

Proposition. *Cyclic n-cocyles are precisely the functionals associated to n-cycles $(\Omega_{\text{univ}}, \int)$ on the universal differential graded algebra over A.*

Lemma. *Let φ be a cyclic n-linear functional. Then*

- $b\varphi$ *is a cyclic $(n+1)$-linear functional, and*
- $b^2\varphi = 0$. □

Definition. Let A be an algebra. The *nth cyclic cohomology group of A* is

$$HC^n(A) = \left\{ \begin{array}{l} \text{cyclic } n\text{-cocyles} \\ \quad \text{modulo cyclic coboundaries.} \end{array} \right\}$$

Proposition. *The formula*

$$\langle \phi, P \rangle = \sum \phi(P_{i_0 i_1}, P_{i_1 i_2}, \ldots, P_{i_n i_0})$$

defines a pairing $HC^{2n}(A) \otimes K_0(A) \to \mathbb{C}$. □

Cyclic Cohomology and Manifolds

For $V^n \subseteq M^d$, oriented, we define

$$\varphi_V(a^0, a^1, \ldots, a^n) = \int_V a^0 da^1 \cdots da^n,$$

where d is the de Rham differential. We obtain maps

geometric n-cycles $\rightarrow$ *closed de Rham currents* $\rightarrow$ *cyclic n-cocyles*

In fact Connes identified $HC^*(C^\infty(M))$ with de Rham homology (details later). Note however that:

- b(n-current) = $(n-1)$-current
- b(cyclic n-cochain) = cyclic $(n+1)$-cocycle
- de Rham currents (not closed) do not determine cyclic cocycles.

So the situation is not altogether straightforward.

Godbillon-Vey Class

Let $A = C^\infty(S^1) \rtimes \Gamma$, where $\Gamma \subseteq \mathrm{Diffeo}^+(S^1)$. Define, for $a^j = \sum_{g \in \Gamma} a^j_g [g] \in A$,

$$\phi(a^0, a^1, a^2) = \sum_{g_0 g_1 g_2 = 1} \int_{S^1} a^0_{g_0} a^1_{g_1} a^2_{g_2} c(g_1, g_2),$$

where

$$c(g_1, g_2) = \log(g_2') d \log(g_1') - \log(g_1') d \log(g_2').$$

This is Connes' Godbillon-Vey cocycle, a cyclic 2-cocycle on A.

Suppose now that $\Gamma = \pi_1(W)$. Form the manifold $M = S^1 \times_\Gamma \widetilde{W}$ and denote by $T_W M$ the codimension 1 bundle of tangent vectors to M which are tangent to W. According to Connes, the cocycle ϕ corresponds to the Godbillon-Vey 3-form

$$\omega = \alpha \wedge d\alpha, \qquad \mathrm{kernel}(\alpha) = T_W M$$

on M.

Let $J_2(S^1) = S^1 \times \mathbb{R}^+ \times \mathbb{R}$. This is the bundle of *2-jets of orientation-preserving diffeomorphisms.* The group $\mathrm{Diffeo}^+(S^1)$ acts on $J_2(S^1)$ by

$$g\colon (t, a, b) \mapsto \left(g(t), g'(t)a, g'(t)b + g''(t)\frac{a^2}{2} \right).$$

(The formula comes from the computation

$$\begin{aligned} & g(t + sa + s^2 b) \\ & = g(t) + sg'(t)a + s^2 \left(g'(t)b + g''(t)\frac{a^2}{2} \right) + o(s^2) \end{aligned}$$

which proves that we get an action.)

Lemma. *The differential 3-form* $\sigma = -\frac{1}{a^3} dt\,da\,db$ *on* $J_2(S^1)$ *is* $\mathrm{Diffeo}^+(S^1)$*-invariant.* □

Lemma. *Suppose that* $\Gamma \subseteq \mathrm{Diffeo}^+(S^1)$ *and that* $\pi_1(W) = \Gamma$*. If* $\omega = \alpha \wedge d\alpha$ *is the Godbillon-Vey class on* $S^1 \times_\Gamma \widetilde{W}$ *then the pullback of* ω *along the map*

$$J_2(S^1) \times_\Gamma \widetilde{W} \xrightarrow[\sim]{} S^1 \times_\Gamma \widetilde{W}$$

is cohomologous to σ*.* □

Hochschild Cohomology

Definition. The *Hochschild cohomology* of A is the cohomology $HH^n(A)$ of the complex

$$\mathrm{Hom}(A, \mathbb{C}) \xrightarrow{b} \mathrm{Hom}(A \otimes A, \mathbb{C}) \xrightarrow{b} \cdots ,$$

where, as before,

$$\begin{aligned} b\varphi(a^0, \ldots, a^{n+1}) = \varphi(a^0a^1, \ldots, a^{n+1}) \\ - \varphi(a^0, a^1a^2, \ldots, a^{n+1}) \\ + \cdots + (-1)^{n+1}\varphi(a^{n+1}a^0, \ldots . a^n), \end{aligned}$$

Example. If $A = C^\infty(M^d)$ then the Hochschild cohomology $HH^n(A)$ is isomorphic to $\Omega_n(M)$, the de Rham currents.[1] Note that if $V \in \Omega_n(M)$ is an n-current then

$$\phi_V(f^0, f^1, \ldots, f^n) = \int_V f^0 df^1 \ldots df^n$$

is a Hochschild n-cocycle.

[1] To be accurate, here HH^* should be defined using the continuous multilinear forms on A.

Proposition. *The de Rham differential on $\Omega_*(M)$ corresponds to the following operator:*

$$B\phi(a^0,\dots,a^n) = \sum_{j=0}^{n}(-1)^{nj}\phi(1,a^j,a^{j+1},\dots,a^{j-1}) + \sum_{j=0}^{n}(-1)^{n(j-1)}\phi(a^j,a^{j+1},\dots,a^{j-1},1). \quad \square$$

To be somewhat more accurate, $B\phi_{V^n} = n\cdot\phi_{\partial V^n}$.

It is therefore reasonable to expect that B will play some role in the description of cyclic cohomology. This expectation is reinforced by the following formula: $B = NB_0$, where

$$B_0\varphi(a^0,\dots,a^n) = \varphi(1,a^0,\dots,a^n) - (-1)^{n+1}\varphi(a^0,\dots,a^n,1)$$

and

$$N\psi(a^0,\dots,a^n) = \sum_{j=0}^{n}(-1)^{nj}\psi(a^j,a^{j+1},\dots,a^{j-1}).$$

Further important properties of B

- The image of B is comprised of cyclic cochains.
- B vanishes on cyclic cochains.
- $B^2 = 0$.
- $Bb + bB = 0$.
- B defines a morphism

$$HH^n(A) \xrightarrow{B} HC^{n-1}(A)$$

and the composition

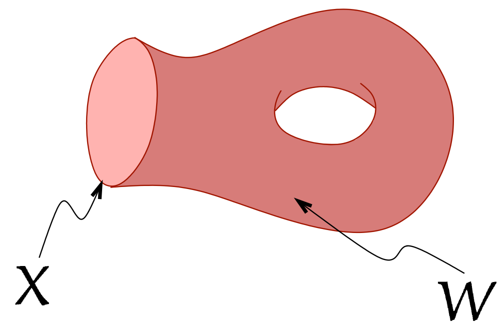
$$HC^n(A) \xrightarrow{I} HH^n(A) \xrightarrow{B} HC^{n-1}(A)$$

is zero. In fact this sequence is exact.

Cycles, Again

Definition. An n-cycle $X = (\Omega_X, \int_X)$ *bounds* if there exists a pair $W = (\Omega_W, \int_W)$, and a surjection $r\colon \Omega_W \to \Omega_X$, such that

$$\int_X r[\omega] = \int_W d\omega$$



Lemma. *If $X = (\Omega_X, \int_X)$ bounds then $c_X(P) = 0$, for all projections P.* □

Remark. This gives a natural context for showing that $[P] \mapsto \int_X \mathrm{Trace}(PdPdP \cdots dPdP)$ is well defined (depends only on $[P] \in K_0(A)$).

Theorem. *A cycle $(\Omega, \int)$ bounds iff its cyclic cohomology class is in the image of the map $B\colon HH^{n+1}(A) \to HC^n(A)$.* □

The S-Operator

Proposition. *The natural product operation on cycles*

$$(\Omega_{A_1}, \int_{A_1}) \times (\Omega_{A_2}, \int_{A_2}) = (\Omega_{A_1} \hat{\otimes} \Omega_{A_2}, \int_{A_1} \hat{\otimes} \int_{A_2})$$

induces

$$HC^{n_1}(A_1) \otimes HC^{n_2}(A_2) \to HC^{n_1+n_2}(A_1 \otimes A_2) \qquad \square$$

Example. $HC^*(\mathbb{C})$ is a polynomial algebra with degree two generator $\varphi(1,1,1) = 1$.

Definition. Denote by

$$S\colon HC^*(A) \to HC^{*+2}(A)$$

the map obtained from the product operation

$$HC^*(\mathbb{C}) \otimes HC^*(A) \to HC^*(A)$$

and the generator of $HC^2(\mathbb{C})$.

Proposition. $\langle \varphi, x \rangle = \langle S\varphi, x \rangle$ *,* $\forall x \in K_0(A)$*.* $\square$

More Remarks on Cycles

Theorem. *A cycle bounds iff its cyclic class is in the kernel of* $S\colon \mathrm{HC}^n(A) \to \mathrm{HC}^{n+2}(A)$ *.*

But if a cycle bounds it is in the image of $B\colon \mathrm{HH}^{n+1}(A) \to \mathrm{HC}^n(A)$. In fact there is an exact sequence

$$\cdots \xrightarrow{I} \mathrm{HH}^{n-1}(A) \xrightarrow{B} \mathrm{HC}^n(A) \xrightarrow{S} \mathrm{HC}^{n+2}(A) \xrightarrow{I} \cdots$$

Example. For $A = C^\infty(M)$ Connes showed[2] that

$$\mathrm{HH}^n(A) = \Omega_n(M)$$
$$\mathrm{HC}^n(A) = Z\Omega_n(M) \oplus H_{n-2}(M) \oplus H_{n-4}(M) \oplus \ldots$$

with the obvious maps in the above sequence.

Problem. Obtain a description of $\mathrm{HC}^n(A)$ in which B, S, the exact sequence, are as transparent as possible.

[2] As before, one works with *continuous* multilinear maps.

The (b,B)-Bicomplex

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow b & & \uparrow b & & \uparrow b & & \uparrow b \\
\mathrm{Hom}(\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A},\mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A},\mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(\mathcal{A}\otimes\mathcal{A},\mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(\mathcal{A}, \\
\uparrow b & & \uparrow b & & \uparrow b & & \\
\mathrm{Hom}(\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A},\mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(\mathcal{A}\otimes\mathcal{A},\mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(\mathcal{A},\mathbb{C}) & & \\
\uparrow b & & \uparrow b & & & & \\
\mathrm{Hom}(\mathcal{A}\otimes\mathcal{A},\mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(\mathcal{A},\mathbb{C}) & & & & \\
\uparrow b & & & & & & \\
\mathrm{Hom}(\mathcal{A},\mathbb{C}) & & & & & &
\end{array}
$$

The (b,B)-Bicomplex, Continued

- The first column, a quotient of the totalized (b, B) bicomplex, is the Hochschild complex.

- The second and higher columns give a subcomplex and a copy of the (b, B)-bicomplex.

- We get the Hochschild-Cyclic long exact sequence from this short exact sequence of complexes.

- The cyclic complex is a subcomplex of the totalized (b, B)-complex, concentrated in the first column.

- **Theorem.** *The inclusion is a quasi-isomorphism.*

The (b,B)-Bicomplex, Continued

A $2n$-cocycle for the (b, B)-bicomplex is a family

$$\Phi = (\Phi_0, \Phi_2, \Phi_4, \ldots, \Phi_{2n})$$

such that $b\Phi_{2j-2} + B\Phi_{2j} = 0$ for all j.

It is conventional to include cyclic cocycles into the (b, B)-bicomplex as follows:

$$\left\{ \begin{array}{l} b\phi_{2k} = 0 \\ \lambda\phi_{2k} = \phi_{2k} \end{array} \right\} \quad \rightsquigarrow \quad \left\{ \begin{array}{l} (0, \ldots, 0, \Phi_{2k}, 0, \ldots) \\ \Phi_{2k} = (-1)^k \dfrac{k!}{(2k)!} \phi_{2k} \end{array} \right\}$$

Theorem. *The formula*

$$\langle \Phi, e \rangle = \sum_{k=0}^{2n} (-1)^k \frac{k!}{(2k)!} \Phi_{2k}(e - \frac{1}{2}, e, e, \ldots, e)$$

defines a pairing $HC^{2n}(A) \otimes K_0(A) \to \mathbb{C}$ *compatible with the previoussly defined pairing between* $K_0(A)$ *and cyclic cocycles.* □

Periodic Cyclic Theory

It is often convenient to periodize HC, as follows.

Definition. The *periodic cyclic cohomology groups* of A are

$$HP^j(A) = \varinjlim_S HC^{j+2k}(A).$$

Theorem. *Let A be an algebra over $\mathbb{C}$ with a multiplicative unit. The periodic cyclic cohomology of A, denoted $HP^*(A)$ is the cohomology of the (direct sum) totalization of the bicomplex*

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow b & & \uparrow b & & \uparrow b \\
\cdots & \xrightarrow{B} & \mathrm{Hom}(A \otimes A \otimes A, \mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(A \otimes A, \mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(A, \mathbb{C}) \\
 & & \uparrow b & & \uparrow b & & \\
\cdots & \xrightarrow{B} & \mathrm{Hom}(A \otimes A, \mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(A, \mathbb{C}) & & \\
 & & \uparrow b & & & & \\
\cdots & \xrightarrow{B} & \mathrm{Hom}(A, \mathbb{C}) & & & &
\end{array}$$

Note. Even cocycles are families $(\Phi_0, \Phi_2, \ldots)$ with $b\Phi_{2j-2} + B\Phi_{2j} = 0$ and with $\Phi_{2j} \equiv 0$ for $j \gg 0$.

Construction of Cyclic Cocycles, I

Definition. Fix an algebra L over $\mathbb{C}$. For $n \geq 0$ denote by $\mathrm{Hom}^n(A, L)$ the vector space of n-linear maps from A to L. Let $\mathrm{Hom}^{**}(A, L)$ be the direct product

$$\mathrm{Hom}^{**}(A, L) = \prod_{n=0}^{\infty} \mathrm{Hom}^n(A, L).$$

Definition. If $\phi \in \mathrm{Hom}^{**}(A, L)$, $\psi \in \mathrm{Hom}^{**}(A, L)$, define

$$\begin{aligned} &\phi \vee \psi(a^1, \dots, a^n) \\ &\qquad = \sum_{p+q=n} \phi(a^1, \dots, a^p)\psi(a^{p+1}, \dots, a^n) \\ &b'\phi(a^1, \dots, a^{n+1}) \\ &\qquad = \sum_{i=1}^{n} (-1)^{i+1} \phi(a^1, \dots, a^i a^{i+1}, \dots, a^{n+1}). \end{aligned}$$

Lemma (Quillen). *The space $\mathrm{Hom}^{**}(A, L)$, so equipped, is a $\mathbb{Z}/2$-graded differential algebra.* □

Definition. Suppose that on L there is a trace $\tau\colon L \to \mathbb{C}$. For $\phi \in \mathrm{Hom}^{**}(A, L)$ define

$$\tau^\natural(\phi)(a^0, \ldots, a^n) = \sum_{i=0}^{n} (-1)^i \tau\big(\phi(a^i, a^{i+1}, \ldots, a^{i-1})\big).$$

Proposition. *The homogeneous parts of* $\tau^\natural(\phi)$ *(as above) are cyclic:*

$$\tau^\natural(\phi)(a^0, \ldots, a^n) = (-1)^n \tau^\natural(\phi)(a^1, \ldots, a^n, a^0).$$

Moreover

$$b\big(\tau^\natural(\phi)\big) = \tau^\natural\big(b'(\phi)\big) \quad \textit{and} \quad \tau^\natural\big([\phi, \psi]_-\big) = 0. \quad \square$$

Corollary. *If* $\phi \in \mathrm{Hom}^{n+1}(A, L)$ *and if*

$$b'\phi = 0 \quad \textit{modulo commutators in } \mathrm{Hom}^{**}(A, L),$$

then $\tau^\natural(\phi)$ *is a cyclic* n*-cocycle.* $\square$

Example. If $\theta \in \mathrm{Hom}^1(A, L)$ is any element and if

$$K = b'\theta + \theta^2$$

then $\tau^\natural(K^n)$ is a cyclic $(2n-1)$-cocycle.

To see this, note that

$$b'K = b'(b'\theta + \theta^2) = b'(\theta^2) = b'\theta \vee \theta - \theta \vee b'\theta = [K, \theta],$$

and since both b' and ad_θ are derivations,

$$b'(K^n) = [K^n, \theta],$$

so that $b'(K^p)$ is a commutator, as required.
Example. If $\theta\colon A \to B$ is a linear map between algebras, if σ is multiplicative modulo an ideal J of B, and if J^n maps to L, we can form $\tau^\natural(K^n)$ (slightly stretching the above analysis). We get

$$K(a^1, a^2) = \theta(a^1 a^2) - \theta(a^1)\theta(a^2)$$

and a cyclic cocycle

$$\begin{aligned}
&\frac{1}{n}\phi(a^0, a^1, \ldots, a^{2n}) \\
&\quad = \tau(K(a^1, a^2)K(a^3, a^4)\ldots K(a^{2n-1}a^{2n})) \\
&\qquad - \tau(K(a^{2n}, a^0)K(a^1, a^2)\ldots K(a^{2n-2}a^{2n-1})).
\end{aligned}$$

This is the cyclic cocycle Connes associates to an extension of algebras.

Preview of Next Lecture

Among other things, we shall discuss a method (also due to Quillen) for constructing cocycles in the (b, B)-bicomplex. This is similar to the method just reviewed, but more complicated.

We shall look at the *JLO cocycle*

$$\Phi_{2n}(a^0, \dots, a^{2n}) =$$
$$\int_{\Sigma^{2n}} \text{Trace}\left(\varepsilon a^0 e^{-t_0\Delta}[D, a^1]e^{-t_0\Delta} \dots e^{-t_1\Delta}[D, a^{2n}]e^{-t_{2n}\Delta}\right)\, dt$$

of a spectral triple (A, H, D) from this perspective. After that we shall turn to the *residue cocycle*

$$\Phi_{2n}(a^0, \dots, a^{2n}) =$$
$$= \sum_{k \geq 0} c_{2n,k}\tau\left(\varepsilon a_0[D, a^1]^{(k_1)} \dots [D, a^{2n}]^{(k_{2n})}\Delta^{-(n+k)}\right)$$

of Connes and Moscovici.