K-Theory and Noncommutative Geometry

Lecture 2
Pseudodifferential Operators

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References for Today

The first two papers deal with the *residue trace* that we shall construct today.

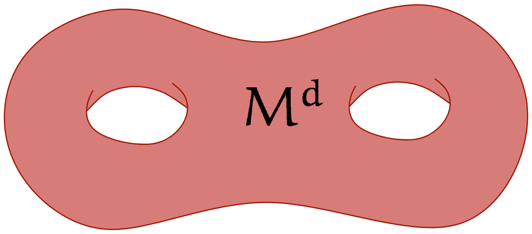
V. Guillemin, *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues*, Advances in Math. **55** (1985), 131–160.

M. Wodzicki, *Noncommutative Residue. I. Fundamentals*, Springer Lecture Notes **1289** (1987), 320-399.

We shall be looking at the residue trace using tools developed in Appendices A and B of the following paper, already cited in the previous lecture.

A. Connes and H. Moscovici, *The local index formula in noncommutative geometry*, Geom. Funct. Anal. **5** (1995), 174–243.

Review



$$\Delta = \nabla^*\nabla$$
$$H^s = \mathrm{domain}(\Delta^{\frac{s}{2}})$$

Definition. Let $H^\infty = \cap_{s\geq 0} H^s$ and $T\colon H^\infty \to H^\infty$.

$$\mathrm{order}_\Delta(T) \leq n \quad \Leftrightarrow \quad T[H^{s+n}] \subseteq H^s, \quad \forall s \geq 0.$$

Definition. $\Psi_n(M,\Delta)$ = operators T on H^∞ which for every $k \in \mathbb{N}$ have the form

$$T = D(I+\Delta)^{-\frac{m}{2}} + R,$$

where D is differential, $\mathrm{order}(D) \leq m+n$, and $\mathrm{order}_\Delta(R) \leq -k$.

Theorem. *The functional*

$$\tau(T) = \frac{1}{2}\mathrm{Res}_{s=0}\Big(\mathrm{Trace}\big(T(I+\Delta)^{-s}\big)\Big)$$

is a trace on the algebra $\Psi(M,\Delta) = \cup_n \Psi_n(M,\Delta)$.

Abstract Pseudodifferential Operators

Let p be a positive integer. Let Δ be a positive, self-adjoint operator on H. Let[1] $H^s = \mathrm{domain}(\Delta^{\frac{s}{p}})$ and let $H^\infty = \cap_{s\geq 0} H^s$.

Let $\mathcal{D}$ be an algebra of operators on H^∞, and assume that $D \in \mathcal{D} \quad \Rightarrow \quad D\Delta, \Delta D \in \mathcal{D}$. Assume that $\mathcal{D}$ is filtered, and write

$$\mathrm{order}_{\mathcal{D}}(D) \leq n \quad \Leftrightarrow \quad D \in \mathcal{D}_n.$$

We shall say Δ is of *Laplace type* for $\mathcal{D}$ if

$$\mathrm{order}_{\Delta}(D) \leq \mathrm{order}_{\mathcal{D}}(D)$$

$$\mathrm{order}_{\mathcal{D}}(\Delta D - D\Delta) \leq \mathrm{order}_{\mathcal{D}}(D) + p - 1.$$

Let $\Psi_n(\mathcal{D}, \Delta)$ be the set of operators T on H^∞ which, for every $k \in \mathbb{N}$, have the form

$$T = D(I + \Delta)^{-\frac{m}{p}} + R,$$

where $D \in \mathcal{D}$, $\mathrm{order}_{\mathcal{D}}(D) \leq m + n$, $\mathrm{order}_{\Delta}(R) \leq -k$.

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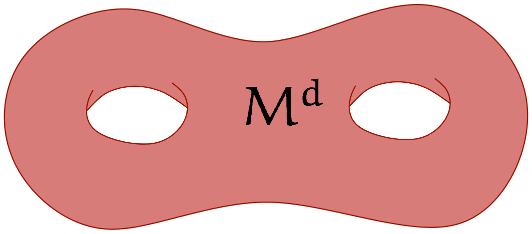
[1] Think of Δ as an order p operator. The case $p = 2$ is most common.

Basic Elliptic Estimate



$W^s(M)$ = Sobolev Space

$\| \ \|_s$ = Sobolev Norm

- An order n differential operator D gives bounded operators $D\colon W^{s+n}(M) \to W^s(M)$.

- $D \sim \sum_{|\alpha|\leq n} f_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}$ is *elliptic of order n* if the function $\sum_{|\alpha|=n} f_\alpha(x)\xi^\alpha$ is bounded below by a multiple of $|\xi|^n$.

- The *basic elliptic estimate* asserts that if D is elliptic of order n then

  $$\|Du\|_s + \|u\|_0 \geq C\|u\|_{s+n}$$

  for all smooth u.

- This implies that for $\Delta = \nabla^*\nabla$ (which is elliptic) the domain of $\Delta^{\frac{s}{2}}$ is $W^s(M)$.

Residue Trace

We shall prove these theorems:

Theorem. $\Psi(\mathcal{D},\Delta) = \cup_n \Psi_n(\mathcal{D},\Delta)$ *is an algebra.*

Let $\Psi^c(\mathcal{D},\Delta)$ be a subalgebra of $\Psi(\mathcal{D},\Delta)$, closed under $T \mapsto \Delta T - T\Delta$.

Theorem. *Assume that for all* $T \in \Psi^c(\mathcal{D},\Delta)$ *the operator* $T(I+\Delta)^{-s}$ *is trace-class when* $s \gg 0$*, and that the function*

$$\zeta(s) = \mathrm{Trace}(T(I+\Delta)^{-s})$$

extends to a meromorphic function on $\mathbb{C}$ *with simple poles. Then the functional*

$$\tau(T) = \frac{1}{p}\,\mathrm{Res}_{s=0}\Big(\mathrm{Trace}(T(I+\Delta)^{-s})\Big)$$

is a trace on $\Psi^c(\mathcal{D},\Delta)$*, meaning that* $\tau(ST) = \tau(TS)$*.*

Remark. We shall discuss analytic continuation a bit later. Here, for now, we just assume it.

Binomial Expansion

Notation. If T is an operator on H^∞ then write

$$T^{(1)} = [\Delta, T] \quad \text{and} \quad T^{(k)} = [\Delta, T^{(k-1)}].$$

Note that $\mathrm{order}_{\mathcal{D}}(D^{(k)}) \leq \mathrm{order}_{\mathcal{D}}(D) + k(p-1)$.

Theorem. *Let $D \in \mathcal{D}_n$ and let $s \in \mathbb{C}$. Then for every $k \in \mathbb{N}$,*

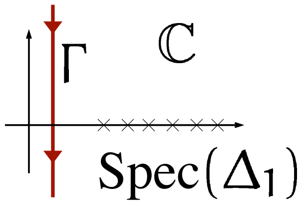
$$\boxed{\begin{aligned} [(I+\Delta)^s, D] = {} & \binom{s}{1} \underset{n+sp-1}{D^{(1)}(I+\Delta)^{s-1}} + \binom{s}{2} \underset{n+sp-2}{D^{(2)}(I+\Delta)^{s-2}} \\ & + \cdots + \binom{s}{k} \underset{n+sp-k}{D^{(k)}(I+\Delta)^{s-k}} + \underset{n+sp-k-1}{R_{k,s}}, \end{aligned}}$$

where

- $\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}$
- $\mathrm{order}_{\Delta}(R_{k,s}) \leq n + sp - k - 1$
- $R_{k,s}$ *is holomorphic in s.*

Remark. *Holomorphic* means holomorphic as a map from half spaces $\mathrm{Re}(s) < \sigma$ into $\mathcal{B}(H, H^{1+k-p\sigma-n})$.

Proof of the Binomial Theorem. Write $\Delta_1 = I + \Delta$. The idea is to use Cauchy's formula:



$$\boxed{\binom{s}{n}\Delta_1^{s-n} = \frac{1}{2\pi i}\int_\Gamma \lambda^s(\lambda - \Delta_1)^{-n-1}\,d\lambda}$$

We'll also use the standard *resolvent identity*

$$\boxed{[X^{-1}, Y] = X^{-1}[Y, X]X^{-1}}$$

We get

$$\begin{aligned}
[\Delta_1^s, D] &= \frac{1}{2\pi i}\int_\Gamma \lambda^s[(\lambda - \Delta_1)^{-1}, D]\,d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma \lambda^s(\lambda - \Delta_1)^{-1}D^{(1)}(\lambda - \Delta_1)^{-1}\,d\lambda \\
&= D^{(1)}\frac{1}{2\pi i}\int_\Gamma \lambda^s(\lambda - \Delta_1)^{-2}\,d\lambda \\
&\qquad + \frac{1}{2\pi i}\int_\Gamma \lambda^s[(\lambda - \Delta_1)^{-1}, D^{(1)}](\lambda - \Delta_1)^{-1}\,d\lambda \\
&= \binom{s}{1}D^{(1)}\Delta_1^{s-1} + R_{1,s}.
\end{aligned}$$

Note that

$$R_{1,s} = \frac{1}{2\pi i}\int_{\Gamma} \lambda^{s}(\lambda - \Delta_1)^{-1} D^{(2)} (\lambda - \Delta_1)^{-2}\, d\lambda.$$

The next step in the iteration is

$$\begin{aligned}
R_{1,s} &= \frac{1}{2\pi i}\int_{\Gamma} \lambda^{s}(\lambda - \Delta_1)^{-1} D^{(2)} (\lambda - \Delta_1)^{-2}\, d\lambda \\
&= D^{(2)} \frac{1}{2\pi i}\int_{\Gamma} \lambda^{s}(\lambda - \Delta_1)^{-3}\, d\lambda \\
&\qquad + \frac{1}{2\pi i}\int_{\Gamma} \lambda^{s}[(\lambda - \Delta_1)^{-1}, D^{(2)}](\lambda - \Delta_1)^{-2}\, d\lambda \\
&= \binom{s}{2} D^{(2)} \Delta_1^{s-2} + R_{2,s},
\end{aligned}$$

where

$$R_{2,s} = \frac{1}{2\pi i}\int_{\Gamma} \lambda^{s}(\lambda - \Delta_1)^{-1} D^{(3)} (\lambda - \Delta_1)^{-3}\, d\lambda.$$

All of these manipulations are valid for $\mathrm{Re}(s) < 0$.

To handle the general case, prove the theorem first for integral s (this is algebra) then write

$$\Delta_1^s = \Delta_1^N \Delta_1^{s-N}$$

and combine the two cases.

Remark. Having proved the binomial theorem for $D \in \mathcal{D}$, exactly the same result now follows for $T \in \Psi(\mathcal{D}, \Delta)$:

$$[(I+\Delta)^s, T] = \binom{s}{1} T^{(1)} (I+\Delta)^{s-1} + \binom{s}{2} T^{(2)} (I+\Delta)^{s-2} + \cdots + \binom{s}{k} T^{(k)} (I+\Delta)^{s-k} + R_{k,s},$$

where

- $\mathrm{order}_\Delta(R_{k,s}) \leq sp + n - k - 1$
- $R_{k,s}$ is holomorphic in s.

This follows immediately from the presentation

$$T = D(I+\Delta)^{-\frac{m}{p}} + \text{low } \Delta\text{-order operator.}$$

Proof that $\Psi(\mathcal{D},\Delta)$ is an algebra. If $T_1, T_2 \in \Psi(\mathcal{D},\Delta)$ then

$$T_1T_2 = D_1(I+\Delta)^{-s_1}D_2(I+\Delta)^{-s_2} + \text{low order operator}$$

Now expand

$$(I+\Delta)^{-s_2}D_2 = D_2(I+\Delta)^{-s_2} + \binom{-s_2}{1}D_2^{(1)}(I+\Delta)^{-s_2-1} + \cdots \qquad \square$$

Proof that the residue is a trace. We have

$$\tau(ST) - \tau(TS) = \frac{1}{p}\mathrm{Res}_{s=0}\Big(\mathrm{Trace}\left(T(I+\Delta)^{-s}S\right) - \mathrm{Trace}\left(TS(I+\Delta)^{-s}\right)\Big).$$

Expand $(I+\Delta)^{-s}S$ to get

$$\tau(ST) - \tau(TS) = \frac{1}{p}\mathrm{Res}_{s=0}\Big(\binom{-s}{1}\mathrm{Trace}\left(TS^{(1)}(I+\Delta)^{-s-1}\right) + \cdots\Big). \qquad \square$$

Meromorphic Continuation

How to show that $\mathrm{Trace}(T\Delta_1^{-s})$ is meromorphic?

Heat kernel expansion

Seeley's Method

Guillemin's method.

Definition. Let $s \in \mathbb{C}$. Denote by $\Psi_s(\mathcal{D}, \Delta)$ the set of operators T on H^∞ which, for every k, may be written

$$T = D(I + \Delta)^{\frac{s-m}{p}} + R,$$

where $D \in \mathcal{D}_m$ and $\mathrm{order}_\Delta(R) \leq -k$.

Theorem. $\Psi_{s_1}(\mathcal{D}, \Delta) \cdot \Psi_{s_2}(\mathcal{D}, \Delta) \subseteq \Psi_{s_1+s_2}(\mathcal{D}, \Delta)$. $\square$

Remark. Note that $\Psi_s(\mathcal{D}, \Delta) = \Psi_0(\mathcal{D}, \Delta) \cdot (I + \Delta)^{\frac{s}{2}}$.

Fix $d \in \mathbb{R}$ and assume that every $T \in \Psi_s^c(\mathcal{D}, \Delta)$ is trace-class if $\mathrm{Re}(s) < -d$.

Guillemin Lemma. *Suppose that for every holomorphic family*[2] *of operators* $T(s) \in \Psi_s^c(\mathcal{D}, \Delta)$ *there are* $U_i \in \Psi_1(\mathcal{D}, \Delta)$, $V_i(s) \in \Psi_s^c(\mathcal{D}, \Delta)$, *and a holomorphic family of operators* $R(s) \in \Psi_s^c(\mathcal{D}, \Delta)$ *such that*

$$(d+s)T(s) = \sum_i [U_i, V_i(s)] + R(s-1).$$

Then $\mathrm{Trace}(T(s))$ *is meromorphic, with simple poles.*

Proof. If $\mathrm{Re}(s) \ll 0$ then

$$\begin{aligned} &\mathrm{Trace}((d+s)T(s)) \\ &\qquad = \mathrm{Trace}(\textstyle\sum_i [U_i, V_i(s)]) + \mathrm{Trace}(R(s-1)) \\ &\qquad = \mathrm{Trace}(R(s-1)) \end{aligned}$$

Hence $\mathrm{Trace}(T(s)) = (d+s)^{-1}\,\mathrm{Trace}(R(s-1))$. $\square$

Remark. The poles of $\mathrm{Trace}(T(-s))$ are located at $d, d-1, d-2, \ldots$.

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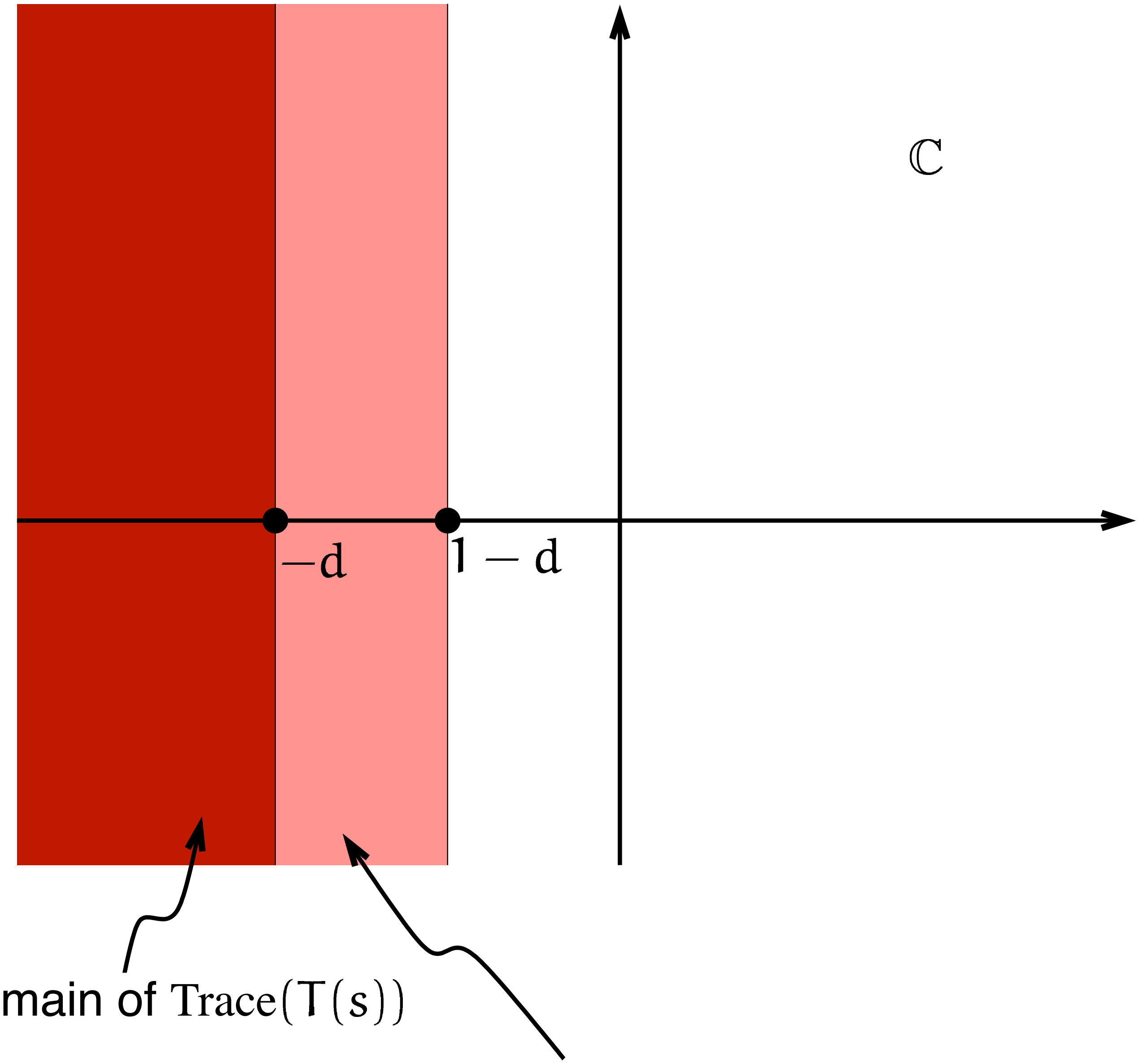
[2] Various definitions of *holomorphic* are possible. Whichever is used, it should imply that $\mathrm{Trace}(T(s))$ is holomorphic when $\mathrm{Re}(s) < -d$.

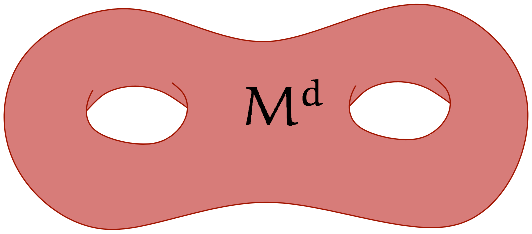
Continuation, Step By Step



Domain of $\mathrm{Trace}(T(s))$

Domain of $\mathrm{Trace}(R(s-1))$

The Manifold Case



$$\Delta = \nabla^* \nabla$$

$$\mathcal{D} = \mathrm{Diff}(M)$$

Lemma. *If* $D \in \mathrm{Diff}_n(M)$*, and if* D *is supported in a coordinate chart, then*

$$\sum_{i=1}^{d} [D, x_i] \frac{\partial}{\partial x_i} = nD - R,$$

where $R \in \mathrm{Diff}_{n-1}(M)$*.* □

Lemma. *If* $D \in \mathrm{Diff}_n(M)$*, and if* D *is supported in a coordinate chart, then*

$$(d+n)D = \sum_{i=1}^{d} \left[D, x_i \frac{\partial}{\partial x_i} \right] - \sum_{i=1}^{d} \left[x_i D, \frac{\partial}{\partial x_i} \right] + R$$

where $R \in \mathrm{Diff}_{n-1}(M)$*.* □

Theorem. *If* $T_s \in \Psi_s(M, \Delta)$ *is holomorphic in* s*, and if* T_s *is supported in a coordinate chart, then*

$$(d+s)T_s = \sum_{i=1}^{d} \left[T_s, x_i \frac{\partial}{\partial x_i} \right] - \sum_{i=1}^{d} \left[x_i T_s, \frac{\partial}{\partial x_i} \right] + R_{s-1}$$

where $R_s \in \Psi_s(M, \Delta)$ *is holomorphic in* s*.* □

Proof. Apply the binomial theorem:

$$\begin{aligned} \sum_{i=1}^{d} [\Delta_1^s, x_i] \frac{\partial}{\partial x_i} &= \sum_{i=1}^{d} s[\Delta_1, x_i] \Delta_1^{s-1} \frac{\partial}{\partial x_i} + \text{lower terms} \\ &= \sum_{i=1}^{d} s[\Delta_1, x_i] \frac{\partial}{\partial x_i} \Delta_1^{s-1} + \text{lower terms} \\ &= 2s\Delta_1 \Delta_1^{s-1} + \text{lower terms} \\ &= 2s\Delta_1^s + \text{lower terms} \end{aligned}$$

Now use

$$T \in \Psi_s(M, \Delta) \quad \Rightarrow \quad \begin{cases} T = D\Delta^{\frac{s+n}{2}} + \text{lower terms} \\ D \in \mathrm{Diff}_n(M). \end{cases}$$

□

Computation of the Residue Trace

The symbol of $T \in \Psi_{-d}(M, \Delta)$ is an order $-d$ homogeneous function on T^*M. Integrating over the unit sphere bundle we obtain the quantity

$$\tau'(T) = \int_{S^*M} \mathrm{Symbol}(T)\,\mathrm{dvol}\,.$$

This is independent of the choice of metric on M^d.

Theorem (Guillemin, Wodzicki). *On $\Psi_{-n}(M, \Delta)$ the residue trace τ is a constant multiple of τ'.*

Proof. Assume M^d is connected and $d > 1$. The de Rham cohomology of S^*M may be computed from the complex of differential forms which are polynomial in each fiber. If $\tau'(T) = 0$ then $\mathrm{Symbol}(T)\,\mathrm{dvol}$ is exact and so $\mathrm{Symbol}(T)$ is a sum of partial derivatives. Hence $\mathrm{Symbol}(T)$ is a sum of Poisson brackets, and so T is a sum of commutators (up to a lower order operator). Hence $\tau(T) = 0$. $\square$

Exercise. The constant depends only on n.

Weyl's Theorem. $\tau(\Delta_1^{-\frac{d}{2}}) = \mathrm{constant} \cdot \mathrm{Vol}(M)$. $\square$

We'll use this in a later lecture:

Theorem. *If* $\Delta' \colon H^\infty \to H^\infty$, *if* Δ' *is positive, and if* $\Delta - \Delta' \in \mathcal{D}_1$ *then*

- Δ' *is of Laplace type for* $\mathcal{D}$.
- $\Psi(\mathcal{D}, \Delta') = \Psi(\mathcal{D}, \Delta)$.
- *If* $T \in \Psi(\mathcal{D}, \Delta)$ *then* $\operatorname{Trace}\bigl(T(I + \Delta')^{-s}\bigr)$ *is meromorphic, with simple poles, assuming this is so for all* $\operatorname{Trace}\bigl(T(I+\Delta)^{-s}\bigr)$, *and the residue traces are equal.*

Proof. The key observation is that if $A = \Delta' - \Delta$ then

$$
\begin{aligned}
&(I+\Delta)^s - (I+\Delta')^s \\
&\qquad = \int_\Gamma \lambda^s (\lambda - \Delta_1)^{-1} A (\lambda - \Delta_1)^{-1}\, d\lambda \\
&\qquad - \int_\Gamma \lambda^s (\lambda - \Delta_1)^{-1} A (\lambda - \Delta_1)^{-1} A (\lambda - \Delta_1)^{-1}\, d\lambda \\
&\qquad + \cdots
\end{aligned}
$$

The trace of each integral may be expanded in zeta functions $\zeta_D(s-j) = \operatorname{Trace}(D\Delta_1^{s-j})$. □

Elliptic Operators

Definition. An (abstract) pseudodifferential operator $T \in \Psi_n(\mathcal{D}, \Delta)$ is *elliptic* if there is a pseudodifferential operator $S \in \Psi_{-n}(\mathcal{D}, \Delta)$ such that

$$ST = I + R_1 \quad \text{and} \quad TS = I + R_2,$$

where $R_1, R_2 \in \Psi_{-1}(\mathcal{D}, \Delta)$.

Remark. If T is elliptic then for every k we can choose S so that $R_1, R_2 \in \Psi_{-k}(\mathcal{D}, \Delta)$.

Example. If $(\mathcal{A}, H, D)$ is a regular spectral triple and if $P \in \mathcal{A}$ is an idempotent then PDP is an elliptic operator on PH (belonging to $\Psi_1(P\mathcal{D}P, P\Delta P)$).

Example. In the manifold case, differential operators

$$D \in \mathrm{Diff}(M) \subseteq \Psi(M, \Delta)$$

which are elliptic in the above sense are elliptic in the standard sense.

Definition of Spectral Triple

Definition. A *spectral triple* is a triple $(\mathcal{A}, H, D)$ consisting of a separable Hilbert space H, an algebra $\mathcal{A}$ of bounded operators on H, and a (typically unbounded) selfadjoint operator D on H, for which:

- the operator $a(I + D^2)^{-1}$ is compact, and
- if $a \in \mathcal{A}$ then the commutator $[D, a] = Da - aD$ is defined on $\mathrm{domain}(D)$ and extends to a bounded operator on H.

Connes proposes that spectral triples will provide an extension of the notion of Riemannian geometric space which is broadly applicable to problems in fundamental physics, number theory, ...

Our Objective: Develop index theory for such a ‘noncommutative geometric space’.

The Standard Example

The basic idea behind spectral triples is that we organize operator theory not around Δ but around a ‘square root’ D, so that $D^2 = \Delta$.

The theory of Dirac-type operators in geometry provides examples in the context of Riemannian manifolds.

- $D = d + d^*$ $\quad D^2 = \nabla^*\nabla$ on forms
- $D =$ Dirac Operator $\quad D^2 = \nabla^*\nabla$ on spinors

Thus on a closed spin manifold M we can take

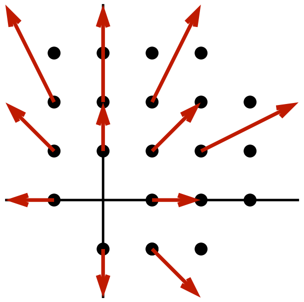
- $A = C^\infty(M)$
- $D =$ Dirac Operator (its self-adjoint extension)
- $H = L^2(M, S)$

A Non-Standard Example

Let

$$\begin{cases} G = \text{simply connected nilpotent group} \\ \Gamma = \text{Lattice in } G \\ V = \Lambda^* \operatorname{Lie}(G) \end{cases}$$

For example, let Γ be the integer Heisenberg group in the real Heisenberg group. Now set



- $A = \mathbb{C}[\Gamma]$
- $H = \ell^2(\Gamma, V)$
- $(Df)(\gamma) = \log(\gamma) \wedge f(\gamma) + \log(\gamma) \vee f(\gamma)$

Remark. If $G = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$ then this triple is isomorphic (via the Fourier transform) to the de Rham operator triple for the torus dual to $\mathbb{Z}^n$.

Regularity

Definition. A spectral triple (A, H, D) is *regular* if there is an algebra B of bounded operators on H such that

- $A \subseteq B$ and $[D, A] \subseteq B$.
- For each $T \in B$ the commutator $[\Delta^{\frac{1}{2}}, T]$ is defined on $\mathrm{domain}(\Delta^{\frac{1}{2}})$ and extends to a bounded operator on H which is again a member of B. Here $\Delta = D^2$.

Definition. Let (A, H, D) be a spectral triple, and assume that $A \cdot H^\infty \subseteq H^\infty$. Denote by $\mathrm{Diff}(A, D)$ the algebra of operators generated by A and D.

Filtration:
$$\begin{cases} A, [D, A] \in \mathrm{Diff}_0(A, D), \quad D \in \mathrm{Diff}_1(A, D) \\ \mathrm{Diff}_i(A, D) \cdot \mathrm{Diff}_j(A, D) \subseteq \mathrm{Diff}_{i+j}(A, D) \\ [\Delta, \mathrm{Diff}_k(A, D)] \subseteq \mathrm{Diff}_{k+1}(A, D) \end{cases}$$

Theorem. *A spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is regular if and only if*

- *each operator* $a \in \mathcal{A}$ *maps* $\mathcal{H}^\infty$ *into itself, and*
- $\Delta = \mathcal{D}^2$ *is of Laplace type for* $\operatorname{Diff}(\mathcal{A}, \mathcal{D})$.

Proof. If Δ is of Laplace type we can form the pseudodifferential operator algebra $\Psi(\mathcal{A}, \mathcal{D})$. Let $\mathcal{B} = \Psi_0(\mathcal{A}, \mathcal{D})$ (as in the definition of regularity).

Conversely, if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is regular then

$$\operatorname{order}_\Delta \leq \operatorname{order}_\mathfrak{D}$$

by induction on the value of $\operatorname{order}_\mathfrak{D}$. Hence Δ is of Laplace type. $\square$

Definition. A spectral triple (A, H, D) is *even* if the Hilbert space H is provided with a $\mathbb{Z}/2$-grading, if the algebra A acts as grading-preserving operators, and if the operator D is grading-reversing.

$$a = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix}, D = \begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix} \quad \text{and} \quad \varepsilon = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Lemma. *If $P \in \Psi_0(A, D)$ is an idempotent operator and if $T \in \Psi(A, D)$ is elliptic of positive order and grading-reversing, then the operator*

$$PTP \colon PH_0 \to PH_1$$

is Fredholm.

We obtain, for a ring $R \subseteq \Psi_0(A, D)$, a map

$$\mathrm{Index}_T \colon K_0(R) \to \mathbb{Z}.$$

Index Problem. Compute this, especially in the case $R = A$ and $T = D$.

The Dixmier Trace

Review: If T is a compact operator and if $\mu_1(T), \mu_2(T), \ldots$ are its singular values then there orthonormal sets $\{v_n\}$ and $\{w_n\}$ in H such that

$$Tv = \sum_n \mu_n(T)\langle v_n, v\rangle w_n$$
$$T^*Tv = \sum_n \mu_n(T)^2\langle v_n, v\rangle v_n.$$

As a result compact operator theory shares much in common with the analysis of sequences.

Inequalities:

$$\begin{cases} \bullet \quad \mu_n(S+T) \le \mu_n(S) + \mu_n(T) \\ \bullet \quad \mu_n(S) + \mu_n(T) \le \mu_{2n}(S+T) \\ \bullet \quad \mu_n(ST) \le \|S\|\mu_n(T) \\ \bullet \quad \mu_n(TS) \le \mu_n(T)\|S\| \end{cases}$$

Example. $\mathcal{L}^1(H) = \left\{T \text{ compact} \mid \sum \mu_n(T) < \infty\right\}$.

Definition. Denote by $\mathcal{L}^{1,\infty}(H)$ the space of compact operators on H for which

$$\boxed{\sup_n n \cdot \mu_n(T) < \infty.}$$

Observe that $\mathcal{L}^{1,\infty}(H)$ is an ideal in $\mathcal{B}(H)$.

Abelian-Tauberian Theorem. *Suppose* T *is compact.*

$$\begin{aligned} \sup_n n \cdot \mu_n(T) = C \quad &\Leftrightarrow \quad \lim_{N\to\infty} \frac{1}{\log(N)} \sum\nolimits_{n\leq N} \mu_n(T) = C \\ &\Leftrightarrow \quad \lim_{s\searrow 1} \frac{1}{s-1} \sum\nolimits_1^\infty \mu_n(T)^s = C. \quad \square \end{aligned}$$

Definition. If $T \in \mathcal{L}^1(H)$ is positive and LIM_ω is a Banach limit then define

$$\mathrm{Tr}_\omega(T) = \mathrm{LIM}_\omega \frac{1}{\log(N)} \sum\nolimits_{n\leq N} \mu_n(T).$$

Theorem (Dixmier). *If* LIM_ω *has the property*

$$\mathrm{LIM}_\omega(\sigma_1, \sigma_2, \sigma_3, \dots) = \mathrm{LIM}_\omega(\sigma_1, \sigma_1, \sigma_2, \sigma_2, \dots)$$

then $\mathrm{Tr}_\omega(T_1 + T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)$. $\square$

Fix a limit LIM_ω as in the theorem.

The theorem states that Tr_ω is additive on the cone of positive operators in $\mathcal{L}^{1,\infty}(H)$: It therefore extends to a linear functional on $\mathcal{L}^{1,\infty}(H)$. It is automatically a trace on $\mathcal{L}^{1,\infty}(H)$. In fact

$$S \in \mathcal{B}(H), T \in \mathcal{L}^{1,\infty}(H) \quad \Rightarrow \quad \mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS).$$

In general Tr_ω depends on the choice of LIM_ω.

Theorem (Connes). *Suppose that* $\Psi(\mathcal{D}, \Delta)$ *is a pseudodifferential operator algebra for which the residue trace is defined. If* $d \in \mathbb{N}$ *and if the zeta function* $\mathrm{Trace}(\Delta_1^{-s})$ *is holomorphic for* $\mathrm{Re}(s) > d$ *then* $\Psi_{-d}(\mathcal{D}, \Delta)$ *is contained in* $\mathcal{L}^{1,\infty}(H)$ *and*

$$\tau(T) = \mathrm{Tr}_\omega(T),$$

for all $T \in \Psi_{-d}(\mathcal{D}, \Delta)$ *and all Dixmier traces* Tr_ω. $\square$

One says, T is *measurable*.

Connes' Character Formula

Theorem. *Let* (A, H, D) *be an even regular spectral triple, for which* D *is invertible and* $\Delta^{-k} \in \mathcal{L}^{1,\infty}(H)$*. The formula*

$$\phi_\omega(a^0, \ldots, a^{2k}) = \frac{1}{2k}\operatorname{Tr}_\omega(\varepsilon a^0[D, a^1][D, a^2] \ldots [D, a^{2k}]\Delta^{-k})$$

defines a Hochschild cocycle on A*. Its value on any Hochschild cycle is the same as that of the cocycle*

$$\phi(a^0, \ldots, a^{2k}) = \frac{1}{2}\operatorname{Trace}(\varepsilon F[F, a^0][F, a^1][F, a^2] \ldots [F, a^{2k}]),$$

where $F = D\Delta^{-\frac{1}{2}}$*. In particular,* ϕ_ω *is independent of* ω*.* $\square$

Corollary. *Suppose that the Hochschild cocycle* ϕ *pairs nontrivially with some Hochschild cycle. Then* $\operatorname{Tr}_\omega(\Delta^{-k}) \neq 0$*.* $\square$

The conclusion that $\operatorname{Tr}_\omega(\Delta^{-k}) \neq 0$ is in effect a lower bound on the eigenvalue growth for the operator Δ (albeit an indirect one).