

K-Homology, Assembly and Rigidity Theorems for Relative Eta Invariants

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- K-homology, elliptic operators and C^* -algebras.
- Geometric K-homology and index theory.
- The assembly map.
- Relative eta invariants.

Joint work with John Roe.

- M = a smooth, closed manifold.
- D = a linear elliptic operator on M
(partial differential, first order).

Atiyah's observation: D determines a class in the K -homology group $K_0(M)$ via the index map

$$\begin{aligned}\text{Index}: K^0(M) &\longrightarrow \mathbb{Z} \\ [E] &\longmapsto \text{Index}(D_E)\end{aligned}$$

... and the families index map

$$\text{Index}: K^0(M \times X) \rightarrow K^0(X).$$

K-homology and elliptic operators, continued

The point: K-homology conceptualizes many constructions in index theory.

Example: The index is induced from the map $\varepsilon : M \longrightarrow \text{pt}$.

Example: The index of a boundary operator is zero:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_1(W, \partial W) & \longrightarrow & K_0(\partial W) & \longrightarrow & K_0(W) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & K_1(\text{pt}, \text{pt}) & \longrightarrow & K_0(\text{pt}) & \xrightarrow{=} & K_0(\text{pt}) \longrightarrow \cdots \end{array}$$

Example: From $\alpha \in K_0(\mathbb{R}^2)$ and $\beta \in K^0(\mathbb{R}^2)$ with $\alpha \cap \beta = 1$ one obtains Bott periodicity.

Atiyah's question: Can K -homology be **defined** using elliptic operators?

Note: if D is an operator on M and $f: M \rightarrow N$, then we obtain

$$f_*[D] \in K_0(N)$$

A good definition should naturally account for this ... which the definition

$$K_0(M) = K^n(T^*M)$$

($\dim(M) = n$) does not.

Analytic cycles for K-homology, continued

Atiyah's suggestion: An analytic cycle for $K_0(X)$ is a bounded *Fredholm* operator F on a Hilbert space H that is equipped with an action of $C(X)$... such that F is *pseudolocal*: $\varphi_1 F \varphi_2$ is a compact operator, if φ_1, φ_2 have disjoint supports.

The point: (1) For such an (F, H) , the families index map

$$\text{Index}: K^0(X \times Y) \rightarrow K^0(Y).$$

is still defined.

(2) $F = D(I + D^*D)^{-1/2}$ = order zero ψ DO is an example.

Analytic cycles for the odd K-homology group

An analytic cycle for $K_1(X)$ is given by a Fredholm operator F , as before, which is now *self-adjoint*.

Compare: $\left\{ \text{Self-adjoint Fredholms} \right\} \sim \Omega(\text{Bott Spectrum})$.

Definition

$K_0^{\text{analytic}}(X)$ = Grothendieck group of homotopy classes of cycles.

Theorem (Brown, Douglas and Fillmore; Kasparov)

The index mappings identify $K_0^{\text{analytic}}(X)$ with the K-homology group $K_0(X)$.

Definition (for later use ...)

$\mathcal{D}(X)$ = commutant modulo compact operators of $\mathcal{C}(X)$ (acting on a single fixed Hilbert space H).

Now let $\mathcal{C}(X)$ = compact operators on H .

Theorem (for later use ...)

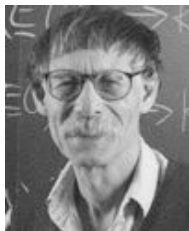
$$K_n^{\text{analytic}}(X) \cong K_{n-1}(\mathcal{D}(X)/\mathcal{C}(X)).$$

Explanation (n odd)

If P is a projection element in $\mathcal{D}(X)/\mathcal{C}(X)$ then the operator

$$F = 2P - I$$

lifts to an analytic cycle for $K_1^{\text{analytic}}(X)$.



Paul Baum

Geometric cycle for $K_n(X)$: (M, E, f)

- M = spin^c -manifold ($n \equiv \dim(M)$).
- E = complex vector bundle on M .
- $f: M \rightarrow X$.

We can form $f_*([D] \cap [E]) \in K_n(X)$.

$\Rightarrow (M, E, f)$ determines a $K_n(X)$ class.

Equivalence relation on geometric cycles.

- **Direct sum/disjoint union** and **Bordism**.
- **Bundle Modification**. P = principal G -bundle over M .
 (S^{2k}, F, ε) = G -equivariant cycle over pt, Index = 1.

$$(M, E, f) \sim (M, E, f) \times_P (S^{2k}, F, \varepsilon)$$

The index theorem

$$\begin{array}{ccc} K_0^{\text{geom}}(X) & \xrightarrow{(M,E,f) \mapsto f_*[D_E]} & K_0^{\text{analytic}}(X) \\ \downarrow h_*^{\text{geom}} & \boxed{h: X \rightarrow Y} & \downarrow h_*^{\text{analytic}} \\ K_0^{\text{geom}}(Y) & \xrightarrow{(M,E,f) \mapsto f_*[D_E]} & K_0^{\text{analytic}}(Y) \\ & \boxed{\varepsilon: X \rightarrow \text{pt}} & \end{array}$$

- $\varepsilon_*^{\text{geom}}$ = topological index
- $\varepsilon_*^{\text{analytic}}$ = analytic index

Poincaré bundle

- $\pi =$ (free) abelian group.
- $\widehat{\pi} =$ Pontrjagin dual.
- $f: X \rightarrow B\pi$.

The *Poincaré line bundle* over $X \times \widehat{\pi}$ is

$$P = (\widehat{X} \times \widehat{\pi} \times \mathbb{C}) / \pi,$$

where the action of π is

$$g \cdot (x, y, z) = (gx, y, \langle g, y \rangle z).$$

We obtain a so-called *dualizing class* $[P] \in K^0(X \times \widehat{\pi})$ and

$$K_n(X) \xrightarrow{\otimes [P]} K^n(\widehat{\pi}).$$

Mishchenko bundle

- $\pi =$ any (torsion-free) group.
- $C^*\pi =$ group C^* -algebra.
- $f: X \rightarrow B\pi$.

The *Mishchenko line bundle* over X is the **flat** line bundle

$$M = (\widehat{X} \times C^*\pi) / \pi.$$

where the action of π is

$$g \cdot (x, y) = (gx, gy).$$

Its fibers are **rank-one free right modules** over $C^*\pi$.

We obtain a class $[M] \in K_0(C(X, C^*\pi))$ (**C^* -algebra K -theory**) and a “dualizing map”

$$K_n(X) \xrightarrow{\otimes [M]} K_n(C^*\pi).$$

The assembly map

Kasparov's *assembly map* is defined to be

$$K_n(B\pi) \xrightarrow{\mu = \otimes[M]} K_n(C^*\pi).$$

Key properties:

- If $D =$ Dirac operator on M and $f: M \rightarrow B\pi$, and if M has positive scalar curvature, then

$$\text{Lichnerowicz} \quad \Rightarrow \quad \mu(f_*[D]) = 0.$$

- If $D =$ signature operator on M and $f: M \rightarrow B\pi$, then

$$\mu(f_*[D]) = \text{oriented homotopy invariant.}$$

The strong Novikov conjecture

Strong Novikov conjecture

The Kasparov assembly map

$$K_n(B\pi) \xrightarrow{\mu} K_n(C^*\pi)$$

is always rationally injective.

Consequences

- Gromov-Lawson-type results . . . for example if M is aspherical and spin, it does not admit a p.s.c. metric.
- The Novikov conjecture.

Other assembly conjectures

Integral strong Novikov conjecture

If π is *torsion-free*, then the Kasparov assembly map is *split injective*.

(Generally false) isomorphism conjecture

If π is *torsion-free*, then the Kasparov assembly map is an *isomorphism*.

Baum-Connes isomorphism conjecture

If π is *torsion-free*, then the *modified* Kasparov assembly map

$$K_n(B\pi) \xrightarrow{\mu} K_n(C_\lambda^*\pi).$$

is an isomorphism.

A consequence of Baum-Connes

Baum-Connes for π
(torsion-free)



Spectrum(x) is connected,
for every $x \in C_\lambda^*(\pi)$



Spectra of elliptic operators on
 π -covering spaces have discrete
component structures

Why? (1) Components in spectra determine projection matrices
over $C_\lambda^*(\pi)$ (functional calculus).

(2) Integrality of the Fredholm index.

The fiber of the assembly map

One can construct “structure groups” $S_n(\pi) \dots$

$$\dots \longrightarrow S_1(\pi) \longrightarrow K_1(B\pi) \xrightarrow{\mu} K_1(C^*\pi) \longrightarrow S_0(\pi) \longrightarrow \dots$$

- Close connections to **surgery exact sequence**.
- Obviously BC $\Leftrightarrow S_*(\pi) \equiv 0$.
- Some (**limited**) use in strong Novikov.
- **Vanishing theorems** in $K_n(C^*\pi)$ made concrete by **constructions** of elements in $S_n(\pi)$.

The analytic surgery exact sequence

Recall we previously defined:

- $H = L^2(X)$, say.
- $\mathcal{D}(X) = C^*$ -algebra of “pseudolocal operators” on H .

Now define, given $f: X \rightarrow B\pi$:

- $H_\pi = L^2(X, M)$
 - = sections of the Mishchenko line bundle
 - = a Hilbert $C^*(\pi)$ -module
 - $\cong L^2(U) \otimes C^*(\pi)$, locally.
- $\mathcal{D}_\pi(X) = C^*$ -algebra of operators on H_π that are locally $T \otimes I$, with $T \in \mathcal{D}(X)$, modulo compact operators.

We get :

$$0 \longrightarrow \mathcal{C}_\pi(X) \longrightarrow \mathcal{D}_\pi(X) \longrightarrow \mathcal{D}(X)/\mathcal{C}(X) \longrightarrow 0$$

The analytic surgery exact sequence, continued

From

$$0 \longrightarrow \mathcal{C}_\pi(X) \longrightarrow \mathcal{D}_\pi(X) \longrightarrow \mathcal{D}(X)/\mathcal{C}(X) \longrightarrow 0$$

we get:

$$\begin{array}{ccccccc} \longrightarrow & K_0(\mathcal{D}_\pi(X)) & \longrightarrow & K_0(\mathcal{D}(X)/\mathcal{C}(X)) & \longrightarrow & K_1(\mathcal{C}_\pi(X)) & \longrightarrow \\ & \downarrow \text{def} & & \downarrow \text{def} & & \downarrow \cong & \\ \longrightarrow & S_1(\pi) & \longrightarrow & K_1(X) & \xrightarrow{\mu} & K_1(\mathcal{C}^*(\pi)) & \longrightarrow \end{array}$$

Example of an analytic structure

Analytic structure = Element of $\mathcal{S}_n(\pi)$.

- M = spin-manifold (odd-dimensional) with p.s.c. metric g
- $f: M \rightarrow B\pi$.
- D_π = Dirac coupled to the (flat) Mishchenko line bundle.

Lichnerowicz $\Rightarrow D_\pi^2 \gg 0$
 \Rightarrow Spectrum(D_π) has gap at $\lambda = 0$.
 \Rightarrow positive spectral projection P_π of D_π lies in $\mathcal{D}_\pi^*(X)$

Definition

$$[M, g] = [P_\pi] \in \mathcal{S}_1(\pi)$$

- $D =$ self-adjoint elliptic operator on M (odd-dimensional).
- The *eta-function* of D is

$$\begin{aligned}\eta_D(s) &= \sum_{\text{Spectrum of } D} \text{sign}(\lambda)|\lambda|^{-s} \\ &= \Gamma\left(\frac{s+1}{2}\right)^{-1} \int_0^\infty \text{Trace}(D e^{-tD^2}) t^{\frac{s-1}{2}} dt.\end{aligned}$$

Theorem

It is meromorphic in $s \in \mathbb{C}$ and *regular* at $s = 0$.

Definition

$\eta(D) = \eta_D(0) = \text{“Trace}(\text{sign}(D))\text{”}$.

Relative eta invariants

- $D =$ self-adjoint elliptic operator on M .
- $f: M \rightarrow B\pi$
- $\sigma_1, \sigma_2: \pi \rightarrow U(N)$.

We can form **twisted operators** D_1 and D_2 using σ_1 and σ_2 .

Definition (Slightly simplified)

$$\begin{aligned}\rho(D, f, \sigma_1, \sigma_2) &= \frac{1}{2} \operatorname{Tr}(\operatorname{sign}(D_1)) - \frac{1}{2} \operatorname{Tr}(\operatorname{sign}(D_2)) \\ &= \frac{1}{2} \eta(D_1) - \frac{1}{2} \eta(D_2) \\ &= \frac{1}{2} \Gamma\left(\frac{1}{2}\right)^{-1} \int_0^\infty \left(\operatorname{Tr}(D_1 e^{-tD_1^2}) - \operatorname{Tr}(D_2 e^{-tD_2^2}) \right) t^{-\frac{1}{2}} dt\end{aligned}$$

Notation

With σ_1 and σ_2 fixed, we'll write $\rho(D, f)$.

Theorem (Atiyah-Patodi-Singer)

If (M, D) is the boundary of (W, Q) , then

$$\text{APS-Index}(Q) = \int_W \text{local expression} - \frac{1}{2}\eta(D).$$

Corollary

If (M, D, f) is a boundary, then $\rho(D, f) \in \mathbb{Z}$.

Rigidity theorems for the relative eta invariant

- $M =$ closed, odd-dimensional spin-manifold with p.s.c. metric.
- $D =$ Dirac operator.
- $f: M \rightarrow B\pi$ and $\sigma_1, \sigma_2: \pi \rightarrow U(N)$.

Theorem (Keswani)

*If π is torsion-free and the Kasparov assembly map is an isomorphism, then the relative eta invariants $\rho(D, f)$ **vanish**.*

Similarly, the relative eta invariants for the signature operators on homotopy equivalent oriented manifolds are equal.

Remark

Related results have been investigated by Mathai, Weinberger, Chang, Schick, Piazza, ...

A theorem of Weinberger

Here is a warm-up theorem, interesting in its own right.

- $M =$ closed, odd-dimensional spin-manifold with p.s.c. metric.
- $D =$ Dirac operator.
- $f: M \rightarrow B\pi$ and $\sigma_1, \sigma_2: \pi \rightarrow U(N)$.

Theorem (Weinberger)

*The relative eta invariants for Dirac operators on p.s.c. spin manifolds are **rational numbers**.*

Remark

There is a similar theorem for signature operators.

Proof using geometric K-homology

- (1) For a geometric cycle (M, E, f) define $\rho(M, E, f) = \rho(D_E, f)$, for some choice of Dirac operator.
- (2) By the APS theorem, different choices of Dirac, or indeed **bordant cycles**, will give the **same** relative eta invariant, **modulo an integer**.
- (3) By an explicit calculation cycles equivalent via bundle modification have the *same* relative eta invariant (for suitable choices of Dirac operators).
- (4) We obtain therefore an **\mathbb{R}/\mathbb{Z} -index map**

$$\text{Index}_{\sigma_1, \sigma_2} : K_1(B\pi) \longrightarrow \mathbb{R}/\mathbb{Z}.$$

- (5) If the assembly map

$$\mu: K_1(B\pi) \longrightarrow K_1(C^*(\pi))$$

was rationally injective, then it would follow that $[D]$ is torsion in the group $K_1(B\pi)$ — and hence that $\text{Index}_{\sigma_1, \sigma_2}[D]$ is torsion in the group \mathbb{R}/\mathbb{Z} .

- (6) But we can replace π by any quotient group through which σ_1 and σ_2 factor — for instance by a *linear* group.
- (7) The assembly map *is* rationally injective for a linear group.

QED

Traces and the surgery exact sequence

Recall the surgery exact sequence is constructed from:

$$0 \longrightarrow \mathcal{C}_\pi(X) \longrightarrow \mathcal{D}_\pi(X) \longrightarrow \mathcal{D}(X)/\mathcal{C}(X) \longrightarrow 0.$$

We want to complete the following diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_0(\mathcal{C}^*\pi) & \longrightarrow & S_1(\pi) & \longrightarrow & K_1(B\pi) \longrightarrow \cdots \\ & & \downarrow \text{Tr}_{\sigma_1} - \text{Tr}_{\sigma_2} & & \downarrow \text{??} & & \downarrow \text{Index}_{\sigma_1, \sigma_2} \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} \longrightarrow 0. \end{array}$$

$$\boxed{\text{Tr}_{\sigma_1, \sigma_2} : S_1(\pi) \longrightarrow \mathbb{R}}$$

The relative trace map via analysis

- $S_1(\pi) = K_0(\mathcal{D}_\pi(X))$
- $T \in \mathcal{D}_\pi(X) \Leftrightarrow T = S \otimes I + K$ locally
- In more detail:
 - (1) $T: L^2(X, M) \rightarrow L^2(X, M)$
 - (2) $L^2(U, M) \cong L^2(U) \otimes C^*(\pi)$ for small $U \subseteq X$
 - (3) $\phi S \psi$ compact if $\text{Supp}(\phi) \cap \text{Supp}(\psi) = \emptyset$.Also, K is compact.

Lemma

Replacing “compact” with “trace class” we obtain a dense subalgebra $\mathcal{D}_\pi^{\text{tracial}}(X)$ on which the formula

$$\text{Tr}_{\sigma_1, \sigma_2}(T) = \text{Tr}_{\sigma_1}(K) - \text{Tr}_{\sigma_2}(K)$$

defines a trace, and hence a map

$$\text{Tr}_{\sigma_1, \sigma_2}: K_0(\mathcal{D}_\pi^{\text{tracial}}(X)) \longrightarrow \mathbb{C}.$$

The relative trace map, continued

After (unfortunately **considerable**) additional work, we obtain from

$$\mathrm{Tr}_{\sigma_1, \sigma_2} : K_0(\mathcal{D}_\pi^{\mathrm{tracial}}(X)) \longrightarrow \mathbb{C}$$

the required relative trace map on $K_0(\mathcal{D}_\pi(X))$:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_0(C^*\pi) & \longrightarrow & S_1(\pi) & \longrightarrow & K_1(B\pi) \longrightarrow \cdots \\ & & \downarrow \mathrm{Tr}_{\sigma_1} - \mathrm{Tr}_{\sigma_2} & & \downarrow \mathrm{Tr}_{\sigma_1, \sigma_2} & & \downarrow \mathrm{Index}_{\sigma_1, \sigma_2} \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} \longrightarrow 0. \end{array}$$

Evaluation of the relative trace on a structure

Lemma

Applied to the structure class of a p.s.c. spin manifold we recover the relative eta invariant of the Dirac operator:

$$\boxed{\mathrm{Tr}_{\sigma_1, \sigma_2}([M, g]) = \rho(D, f).}$$

If $P_\pi =$ positive spectral projection of D_π , then

$$P_\pi = \frac{1}{2}(\mathrm{sign}(D_\pi) + I).$$

Therefore

$$\begin{aligned}\mathrm{Tr}_{\sigma_1, \sigma_2}(P) &= \text{“}\mathrm{Tr}_{\sigma_1}(P) - \mathrm{Tr}_{\sigma_2}(P)\text{”} \\ &= \frac{1}{2} \mathrm{Tr}_{\sigma_1}(D_\pi) - \frac{1}{2} \mathrm{Tr}_{\sigma_2}(D_\pi) = \rho(D, f).\end{aligned}$$

as required.

- $D =$ Dirac on p.s.c. spin-manifold.
- $f: M \rightarrow B\pi$.

Theorem (Keswani)

If the Kasparov assembly map is an isomorphism, then the relative eta invariants of D are zero.

For if the assembly map is an isomorphism, then the structure group $S_1(\pi)$ is zero.

