

# $C^*$ -Algebras and Group Representations

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## Summary

Mackey pointed out an analogy between irreducible representations of a semisimple group and irreducible representations of its Cartan motion group.

Can  $C^*$ -algebras and noncommutative geometry cast new light on Mackey's analogy?

- Some more or less historical remarks.
- The Connes-Kasparov conjecture.
- The Mackey analogy.
- Results and perspectives.

# Groups and Convolution Algebras

- $G$  = a Lie group.
- $C_c^\infty(G)$  = *convolution algebra* of functions on  $G$ .

$$f_1 * f_2(g) = \int_G f_1(h) f_2(h^{-1}g) dh.$$

Under favorable circumstances a group representation  $\pi: G \rightarrow \text{Aut}(V)$  induces an algebra representation:

$$\pi: C_c^\infty(G) \rightarrow \text{End}(V)$$

$$\pi(f)v = \int_G f(g) \pi(g)v dg.$$

Under favorable circumstances the reverse is true as well.

# Compact Groups

... but why bother with the convolution algebra?

One answer: it is effective for compact groups.

- $G$  = a compact Lie group.
- $\sigma: G \rightarrow \text{Aut}(V)$  irreducible representation.
- $\rho(g) = \dim(\sigma) \text{Trace}(\sigma(g^{-1}))$ .

## Theorem (Peter-Weyl)

*If  $G$  is compact, then  $C^\infty(G) \approx \bigoplus_\sigma \text{End}(\sigma)$ .*

From this, Weyl was able to determine the irreducible representations of compact groups.

To study *unitary* representations we might consider this:

## Definition

$C^*(G)$  = Completion of  $C_c^\infty(G)$  in the norm  $\|f\| = \sup_\pi \|\pi(f)\|$ .

It is a  *$C^*$ -algebra*.

## Theorem (Gelfand and Naimark)

*If  $A$  is a commutative  $C^*$ -algebra, then  $A \cong C_0(\hat{A})$ .*

## Corollary

*For abelian groups one has  $C^*(G) \cong C_0(\hat{G})$ .*

The unitary dual of  $G$  may be topologized using convergence of *matrix coefficient functions*:

$$\phi(g) = \langle v, \pi(g)v \rangle.$$

This turns out to be the same as a *hull-kernel topology* on the dual of  $C^*(G)$ :

- $J = \text{ideal in } C^*(G)$ .
- $\{ \pi \mid \pi[J] \neq 0 \}$  = open subset of the dual.

We shall need the *reduced group  $C^*$ -algebra* later . . .

## Definition

The *reduced group  $C^*$ -algebra*  $C_\lambda^*(G)$  is the image of  $C^*(G)$  in the algebra of bounded operators on  $L^2(G)$  under the left regular representation.

It is a quotient of  $C^*(G)$ . So its irreducible representations constitute a closed subset of the unitary dual, called the *reduced or tempered dual*  $\widehat{G}_\lambda$ .

The representations in the reduced dual are those that contribute to the regular representation.

# The Role of $C^*$ -Algebras

- $C^*$ -algebras ought to be useful when the topology of the dual plays an important role.
- There is a natural notion of **Morita equivalence**, and Morita equivalent  $C^*$ -algebras have equivalent categories of representations.
- Example: If  $G$  is compact, then

$$C^*(G) \approx \bigoplus \text{End}(\sigma) \underset{\text{Morita}}{\sim} C_0(\widehat{G}).$$

So Morita equivalence can make clear the parameters describing representations.

# Decomposition of Representations

## Key Early Problem (from 1950 on): Plancherel Formula

Decompose the regular representation on  $L^2(G)$  into irreducible representations.

### Examples

- Compact groups:  $f(e) = \sum_{\sigma} \dim(\sigma) \text{Trace}(\sigma(f))$ .
- Abelian groups:  $f(e) = \int_{\widehat{G}} \widehat{f}(\xi) d\xi$ .

### Two Problems

1. Is there, in fact, a natural decomposition at all?
2. If so, describe it explicitly.

# Admissible Representations

Let  $K$  be a maximal compact subgroup of  $G$ .

## Conjecture of Mautner

If  $G$  is a semisimple group (with finite center) then *every  $K$ -isotypical subspace of every irreducible unitary representation of  $G$  is finite dimensional*. In other words, every irreducible unitary representation of  $G$  is *admissible*.

Admissibility makes it possible to apply the direct integral *decomposition theory* of von Neumann, and so obtain an abstract Plancherel formula. This settles Problem 1.

Mautner's conjecture was proved by Harish-Chandra and Godement, in different ways . . .

The following gives a hint about proving Mautner's conjecture.

- $G =$  semisimple group.
- $K =$  maximal compact subgroup.

## Lemma

*The algebra generated by the  $K$ -bi-invariant functions on  $G$  is abelian, and so its irreducible representations are one-dimensional.* □

Notice that the  $C^*$ -algebra acts on the space of  $K$ -fixed vectors in any unitary representation of  $G \dots$

# Godement's Approach to the Conjecture

- $\sigma =$  irreducible representation of  $K$ .
- $p(k) = \dim(\sigma) \text{Trace}(\sigma(k^{-1}))$ .

Definition (of the local Hecke algebra associated with  $\sigma$ )

$$A^*(G, \sigma) = C^*(G)^K \cap C^*(G)p.$$

## Lemma

*There is an equivalence of categories*

$$\left\{ \begin{array}{l} \text{Unitary rep'ns of } G \text{ generated by} \\ \text{their } \sigma\text{-isotypical subspaces} \end{array} \right\} \approx \left\{ \text{Rep'ns of } A^*(G, \sigma) \right\}.$$

By studying finite-dimensional representations, Godement showed that  $A^*(G, \sigma)$  satisfies suitable **polynomial identities**.

# Harish-Chandra's Approach to the Conjecture

- $U(\mathfrak{g}) =$  *universal enveloping algebra* of (the complexification of)  $\mathfrak{g} = \text{Lie}(G)$ .

Definition (Harish-Chandra, Lepowsky, et al)

The *local Hecke algebra* associated to  $\sigma$  is

$$A(\mathfrak{g}, \sigma) = [U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \text{End}(\sigma)]^K.$$

It is an algebra with multiplication given by the formula

$$(S_1 \otimes T_1)(S_2 \otimes T_2) = S_1 S_2 \otimes T_2 T_1.$$

Lemma

If  $G$  acts on  $W$ , then  $A(\mathfrak{g}, \sigma)$  acts on  $\text{Hom}_K(\sigma, W)$  as follows:

$$(S \otimes T) \cdot L = SLT.$$

# Harish-Chandra's Approach, Continued

The functor from representations of  $G$  to representations of  $A(\mathfrak{g}, \sigma)$  is **not** an equivalence. But it has natural left and right adjoints and it induces a **bijection**

$$\left\{ \begin{array}{l} \text{Irreducible representations of } G \text{ with} \\ \text{non-zero } \sigma\text{-isotypical subspaces} \end{array} \right\} \approx \left\{ \text{Irreducible representations of } A(\mathfrak{g}, \sigma) \right\}.$$

## Theorem (Harish-Chandra)

*The algebra*

$$A(\mathfrak{g}, \sigma) = [U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \text{End}(\sigma)]^K$$

*is finitely generated as a module over the center of the enveloping algebra.*

*Harish-Chandra has obtained deep properties [of spherical functions] using Lie algebra methods extensively.*

*The author makes an attempt to derive some properties . . . without using the Lie algebra . . . but only a few of the results of Harish-Chandra are obtained and some of them in a significantly weaker form.*

*It is a very worthwhile problem to study spherical functions by quite general and purely integral methods, but whether the deeper properties can be so obtained remains an open question.*

Mautner  
Review of Godement's 1952 paper

## Definition

An irreducible unitary representation of  $G$  is a *discrete series representation* if it can be realized as a summand of the regular representation of  $G$  on  $L^2(G)$ .

## Work of Harish-Chandra:

*Parametrization* of the discrete series.

## Work of Langlands, Schmid, Parthasarathy, Atiyah

Geometric *construction* of the discrete series.

## Comment:

The discrete series occur as isolated points in the reduced dual, and so should be accessible by  $C^*$ -algebra methods.

# Index Theory and the Discrete Series

Let  $M = G/K$ . Under an orientation hypothesis on  $M$ , there is a *Dirac induction map*

$$\text{D-Ind}: \langle \text{representations of } K \rangle \longrightarrow \langle \text{representations of } G \rangle.$$

Given a representation of  $K$  on  $V$ , let

$$E = G \times_K (V \otimes S).$$

Here  $S$  is a “spinor representation” of  $K$ .

An equivariant Dirac operator  $D$  acts on the sections of  $E$ , and we define

$$\text{D-Ind}(V) = \text{Kernel}_{L^2}(D).$$

## Theorem (Atiyah and Schmid)

*The discrete series are parametrized by “nonsingular” irreducible  $V$  via Dirac induction.*

Dirac induction gives rise to a map

$$\text{D-Ind}: R(K) \longrightarrow K(C_\lambda^*(G)).$$

It accounts for the representations of  $K$  that are “singular” in the theory of the discrete series . . .

## Connes-Kasparov Conjecture

*The Dirac induction map in  $C^*$ -algebra  $K$ -theory is an isomorphism.*

This is now viewed as part of the *Baum-Connes conjecture*.

# The Connes-Kasparov Conjecture

The Connes-Kasparov conjecture is now proved (**twice**).

*Proof of Wassermann.* Via representation theory (explicit computation of the reduced  $C^*$ -algebra and Dirac induction).

*Proof of Lafforgue.* Via  $KK$ -theory (using a sophisticated generalization of Bott periodicity).

Lafforgue's argument shows that the discrete series are parametrized by a subset of the nonsingular part of dual of  $K$ .

The index theorem and further representation theory show that the subset is all of the nonsingular part.

# Contraction of a Lie Group

- $G =$  Lie group.
- $H =$  closed subgroup.

## Definition

The *contraction of  $G$  along  $H$*  is the Lie group  $H \times \text{Lie}(G)/\text{Lie}(H)$ .

The contraction of  $G$  along  $H$  is a first-order, or linear, approximation of  $G$  in a neighborhood of  $H$ .

## Example

- $G = SO(3)$  (the rotation group of a sphere)
- $H = SO(2)$ .

The contraction of  $G$  along  $H$  is the group  $SO(2) \times \mathbb{R}^2$  of rigid motions of the plane.

## Example

- $X$  = hyperbolic 3-space.
- $G$  = (orientation preserving) isometries of  $X = PSL(2, \mathbb{C})$ .
- $K$  = isotropy group of a point =  $PSU(2) = SO(3)$ .

At small scales the space  $X$  is nearly Euclidean.

The contraction of  $G$  along  $K$  is the group of rigid motions of 3-dimensional Euclidean space.

# George Mackey, 1916-2006



# Mackey Analogy

- $G$  = semisimple Lie group (with finite center).
- $K$  = maximal compact subgroup.
- $G_0$  = contraction of  $G$  along  $K$ .

*“. . . the physical interpretation suggests that there ought to exist a ‘natural’ one to one correspondence between almost all of the unitary representations of  $G_0$  and almost all the unitary representations of  $G$ —in spite of the rather different algebraic structures of these groups.”*

George Mackey, 1975

# Mackey Analogy, Continued

## Mackey's description of the dual of $K \ltimes V$

$$\varphi \in \widehat{V}, \tau \in \widehat{K}_\varphi \longrightarrow \text{Ind}_{K_\varphi \ltimes V}^{K \ltimes V}(\tau \times \varphi) \in \widehat{K \ltimes V}$$
$$\widehat{K \ltimes V} \cong \left( \bigsqcup_{\varphi \in \widehat{V}} \widehat{K}_\varphi \right) / K$$

## An example of the analogy

$G =$  complex semisimple,  $G = KB$ ,  $B = MAN$ .

$$\varphi \in \widehat{A}, \sigma \in \widehat{M} \longrightarrow \text{Ind}_B^G(\sigma \times \varphi) \in \widehat{G}_\lambda.$$

Typically  $K_\varphi = M$ . Note that  $M \ltimes V$  is to  $B$  as  $K \ltimes V$  is to  $G$ .

Also

$$\text{Ind}_B^G(\sigma \times \varphi) = \text{Ind}_{M \ltimes V}^{K \ltimes V}(\sigma \times \varphi),$$

as representations of  $K$ .

## Complex semisimple groups: conclusion

“Typical” tempered representations of  $G$  and of the contracted group  $G_0 = K \ltimes V$  agree, both in parametrization and in basic form.

*We have not yet ventured to formulate a precise conjecture . . . However we feel sure that some such result exists . . .*

Mackey, 1975

# Smooth Family of Lie Groups

There is a **smooth family of Lie groups** interpolating between the groups  $G_0$  and  $G$ .

- $G_t = G$  for all  $t \neq 0$ .
- If  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , then for any  $k \in K$  and  $X \in \mathfrak{p}$ , the family

$$\begin{cases} 0 \mapsto (k, X) \\ t \mapsto k \exp(tX) \end{cases}$$

is a smooth section.

# Reformulation of the Connes-Kasparov Isomorphism

Associated to the smooth family of Lie groups  $\{G_t\}_{t \in \mathbb{R}}$  is a continuous field of group  $C^*$ -algebras  $\{C_\lambda^*(G_t)\}_{t \in \mathbb{R}}$ .

The Connes-Kasparov isomorphism can be reformulated using this continuous field as follows:

## Theorem (Connes and Higson)

*The continuous field of group  $C^*$ -algebras  $\{C_\lambda^*(G_t)\}_{t \in \mathbb{R}}$  has **constant**  $K$ -theory.*

In some sense, this makes the Mackey analogy precise.

Can this precise cohomological statement be reconciled with Mackey's representation-by-representation analogy?

# Parameters for Representations

- $G$  = complex semisimple group.
- $\widehat{G}_\lambda = (\widehat{M} \times \widehat{A}) / W$  (principal series construction).
- $G_0$  = contracted group.
- $\widehat{G}_0 = (\bigsqcup_{\varphi \in \widehat{V}} \widehat{K}_\varphi) / K$  (Mackey machine).
- Up to conjugacy one has  $\varphi \in \widehat{A}$ , and then

$$M \subseteq K_\varphi \subseteq K.$$

Moreover  $K_\varphi$  is *connected*.

- By the Cartan-Weyl highest weight theory,

$$\widehat{G}_0 = \left( \bigsqcup_{\varphi \in \widehat{V}} \widehat{K}_\varphi \right) / K = \left( \bigsqcup_{\varphi \in \widehat{A}} \widehat{M} / W_\varphi \right) / W = (\widehat{M} \times \widehat{A}) / W = \widehat{G}_\lambda.$$

# Topological Structure of the Duals

The bijection is **not** a homeomorphism and does not by itself explain the Connes-Kasparov isomorphism. However ...

- $A_\lambda^*(G, \sigma) \subseteq C_\lambda^*(G)$ , local Hecke algebra.
- $\widehat{K}$  is ordered by highest weights.
- $J_\lambda^*(G, \sigma) =$  ideal in  $C_\lambda^*(G)$  corresponding to representations with an isotypical summand indexed by some  $\sigma' < \sigma$ .

## Theorem

*The continuous field  $\{A_\lambda^*(G_t, \sigma) / (A_\lambda^*(G_t, \sigma) \cap J_\lambda^*(G_t, \sigma))\}$  is isomorphic to a constant field of commutative  $C^*$ -algebras.*

## Theorem (again)

*The continuous field  $\{A_\lambda^*(G_t, \sigma) / (A_\lambda^*(G_t, \sigma) \cap J_\lambda^*(G_t, \sigma))\}$  is isomorphic to a constant field of commutative  $C^*$ -algebras.*

The  $C^*$ -algebra  $A_\lambda^*(G, \sigma)$  is Morita equivalent to the ideal in  $C_\lambda^*(G)$  associated to representations that have a non-zero  $\sigma$ -isotypical summand.

## Conclusions

1. The field  $\{C_\lambda^*(G_t)\}$  is assembled from constant fields of commutative  $C^*$ -algebras by Morita equivalences, extensions, and direct limits.
2. The duals of  $G$ , complex semisimple, and its contraction  $G_0$  are assembled from the same constituent pieces, consisting of representations with a given lowest  $K$ -type.

# Local Hecke Algebras

A second look at the algebraic local Hecke algebra:

$$A(\mathfrak{g}, \sigma) = [U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \text{End}(\sigma)]^K.$$

This algebra is filtered by degree in  $U(\mathfrak{g})$  and it is easy to compute the **associated graded algebra**:

$$\text{gr } A(\mathfrak{g}, \sigma) = [S(\mathfrak{p}) \otimes \text{End}(\sigma^\diamond)]^K.$$

This is precisely the local Hecke algebra  $A(\mathfrak{g}_0, \sigma)$  for the contracted group  $G_0$ . It follows for example that

$$HP_*(A(\mathfrak{g}_0, \sigma)) \cong HP_*(A(\mathfrak{g}, \sigma)),$$

which is a sort of (not so interesting) algebraic version of the Connes-Kasparov isomorphism.

# Complex Semisimple Groups

If  $G$  is complex semisimple, then

$$\mathfrak{g}_{\mathbb{C}} \cong \bar{\mathfrak{g}} \oplus \mathfrak{g} \cong \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}.$$

As a result, we get a simplified formula for the local Hecke algebra:

$$A(\mathfrak{g}, \sigma) = [U(\mathfrak{k}) \otimes \text{End}(\sigma^{\diamond})]^K.$$

Similarly, for the contracted group  $G_0$  we get

$$A(\mathfrak{g}_0, \sigma) = [S(\mathfrak{k}) \otimes \text{End}(\sigma^{\diamond})]^K.$$

These are Kirillov's quantum and classical **family algebras**.

# Langlands Classification

For each  $\nu \in \text{Hom}(MA, \mathbb{C}^\times)$  there is a not-necessarily unitary principal series representation

$$\text{Ind}_B^G \nu$$

of the complex semisimple group  $G$ .

The lowest  $K$ -type of  $\text{Ind}_B^G \nu$  is the representation with highest weight  $\nu|_M$ . It has multiplicity one and therefore it determines a unique irreducible subquotient

$$\Lambda(\nu) \ll \text{Ind}_B^G \nu.$$

## Theorem (Zhelobenko)

*The correspondence  $\nu \leftrightarrow \Lambda(\nu)$  determines a bijection*

$$\text{Hom}(MA, \mathbb{C}^\times) / W \cong \langle \text{nonunitary dual of } G \rangle.$$

# Mackey Analogy for Nonunitary Representations

Let  $G$  be complex semisimple and let  $G_0 = K \ltimes V$  be its contraction.

The parameters for the irreducible, not necessarily unitary representations of any semidirect product  $K \ltimes V$  may be determined using local Hecke algebras:

## Theorem (Rader)

$$\langle \text{nonunitary dual of } K \ltimes V \rangle \cong \left( \bigsqcup_{\varphi \in \text{Hom}(V, \mathbb{C}^\times)} \widehat{K}_\varphi \right) / K.$$

Therefore as before, we obtain a Mackey bijection

$$\langle \text{nonunitary dual of } G_0 \rangle \cong \langle \text{nonunitary dual of } G \rangle.$$