Metric geometry and local index formulae

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Abstract

In this last talk we'll summarize some of Connes' ideas about the metric aspect of noncommutative geometry and associated local index formulae. Main Reference: Connes and Moscovici, *The local index formula in noncommutative geometry*.

Metrics and noncommutative geometry

What does it mean to give a metric on a noncommutative space described by an algebra A?

According to Connes, this data is given by a *spectral triple* (a.k.a. unbounded Fredholm module), that is, a representation $\rho: A \to \mathfrak{B}(H)$ together with an unbounded selfadjoint operator D on H, with compact resolvent, such that $[D, \rho(a)]$ is bounded for a dense set of $a \in A$.

Why? Consider the example of a Riemannian manifold M. Then one can recover the distance between two points x, y of M as

$$\sup\{|f(x) - f(y)| : \|[D, \rho(f)]\| \leq 1\}.$$

Passing from D to $F = D|D|^{-1}$ should be thought of as passing from metric to conformal structure.

Summability

An unbounded Fredholm module is p-*summable* if the resolvents $(D \pm i)^{-1}$ belong to the Schatten ideal $\mathfrak{L}^p$. (This implies the corresponding condition for F.)
Exercise: The unbounded Fredholm module defined by the length function on a group is finitely summable if and only if the group has polynomial growth.

We will focus attention on p-summable modules. But one should not think that this condition is universal. In fact one has:

Theorem 1. *If Γ is a nonamenable group then there are no finitely summable* unbounded *Fredholm modules on $C^*_r(\Gamma)$.*

There do exist bounded p-summable Fredholm modules on $C^*_r(\Gamma)$ even for some property T groups Γ.
Question: Are there examples in rank > 1?

Constructing the character

For a finitely summable *bounded* Fredholm module we have seen how to construct a cyclic character:

$$\tau_q(a_0, a_1, \ldots, a_q) = \mathrm{Tr}'(a_0[F, a_1] \cdots [F, a_q]).$$

Recall that this construction produces many characters for the same Fredholm module, which are related by the periodicity operator $S\colon HC^q(A) \to HC^{q+2}(A)$. Ask: What is the *lowest* dimension in which the cyclic character exists?

By Connes' exact sequence to ask whether τ_q can be 'desuspended' is the same as to ask whether the image $I(\tau_q)$ in Hochschild cohomology $HH^q(A)$ is zero. How to calculate this image?

The answer involves the *Dixmier trace.*

The Dixmier trace

Let H be a Hilbert space. The *Macaev ideal* $\mathfrak{M}$ is the collection of all compact operators T on H whose characteristic values μ_i (the eigenvalues of $|T|$) satisfy

$$\sup_N \frac{1}{\log N} \sum_{i=1}^{N} \mu_i < \infty.$$

Define the *Dixmier trace*

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\log N} \sum_{i=1}^{N} \lambda_i$$

where ω is a (suitable?) ultrafilter on $\mathbb{N}$. This formula defines a *linear* (Surprise!) functional on $\mathfrak{M}$, which is clearly a trace. Moreover it annihilates all operators in the ordinary trace class $\mathfrak{L}^1$.

(Connes denotes $\mathfrak{M}$ by $\mathfrak{L}^{1,\infty}$.)

Wodzicki residue

One can show that a pseudodifferential operator of order $-n$ (for instance $(1 + D^2)^{-n/2}$) on an n-dimensional manifold belongs to the ideal $\mathfrak{M}$. (Look at the circle for an example.)

Theorem 2. *Let P be such a pseudodifferential operator with principal symbol $\sigma(x, \xi)$ (a homogeneous function of degree $-n$ on T^*M.) Then (for any choice of the ultrafilter ω),*

$$\mathrm{Tr}_\omega(P) = c_n \int_{S^*M} \sigma dv.$$

(In other words, the trace depends only on the principal symbol. The right hand side is the *Wodzicki residue*; Wodzicki extended it to a trace on the algebra of *all* pseudodifferential operators. It is the residue at $s = 0$ of the meromorphic function $s \mapsto \mathrm{Tr}(P\Delta^{-s/2})$.)

Proof Vanishing on $\mathfrak{L}^1$ and symmetry.

Key Point: The symbol is a local invariant.

Classical Limit v. Dixmier Trace

Let F be the Fredholm module defined by the Dirac operator D over a spin manifold M^n. It is p-summable for all $p > n$ (as we saw above), so it has a character in $HC^n(C^\infty(M))$.

Recall from lecture 4 that

$$HC^n(C^\infty(M)) = Z_n(M) \oplus H_{n-2}(M) \oplus H_{n-4}(M) \oplus \cdots.$$

Theorem 3. *The character of the Dirac module has top-dimensional component equal to the current* $(f_0, \ldots, f_n) = \int_M f_0 df_1 \cdots df_n$*, and the homology components are those of the Poincaré dual of the* $\widehat{\mathcal{A}}$*-genus.*

The proof uses the classical limit. Connes' idea: Replace this by the Dixmier trace.

The residue theorem for the cyclic character

Let (ρ, H, D) define an unbounded Fredholm module over A, and suppose that this module is $\mathfrak{L}^{n,infty}$ summable (in the obvious sense).

Then for $a_0, \ldots, a_n \in \mathcal{A}$ the operator $a_0[D, a_1] \cdots [D, a_n]|D|^{-n}$ belongs to the Macaev ideal $\mathfrak{M}$, and

$$\phi_\omega(a_0, \ldots, a_n) = \mathrm{Tr}_\omega\big(a_0[D, a_1] \cdots [D, a_n]|D|^{-n}\big)$$

defines a Hochschild cocycle on $\mathcal{A}$.

Theorem 4. *This Hochschild cocycle 'is' the image of the n-dimensional cyclic character τ_n under the map $I\colon HC^n(\mathcal{A}) \to HH^n(\mathcal{A})$. Consequently, if the cohomology class of ϕ_ω is not zero, then τ_n cannot be desuspended — it does not belong to the image of S.*

Proof Exercise?! An easier exercise — compute in the manifold example.

Dimension spectrum

To get a more refined formula one needs to incorporate lower order terms in asymptotic expansion.

Definition 1. *The* dimension spectrum *of a spectral triple (if it exists) is a discrete subset* $\Sigma \subseteq \mathbb{C}$ *with the property that all the zeta functions*

$$\zeta_b(s) = \mathrm{Tr}(b|D|^{-s}),$$

where b *is a useful operator, continue analytically to* $\mathbb{C} \setminus \Sigma$*.*

(The 'useful' operators are just all the things like $[|D|, a]$ which are going to appear in the formulae.) We'll assume that the continuations have simple poles on Σ; there is a more elaborate version of the theory which allows for multiple poles.

Example: The Hilsum-Skandalis hypoelliptic operator corresponding to a triangular structure.

Extended Dixmier trace

One can define a notion of *abstract pseudodifferential operator*, given by expansions like

$$b_q|D|^q + b_{q-1}|D|^{q-1} + \cdots$$

where the b's are useful operators. For any such pseudodifferential operator P, the function $\zeta_P(s) = \mathrm{Tr}(P|D|^{-s})$ is meromorphic on $\mathbb{C}$, and the residue $\chi_0(P)$ of $\zeta_P(s)$ at zero is an extension of the Dixmier trace on $\mathfrak{M}$ to *all* pseudodifferential operators — exactly as in Wodzicki's work.

Remark: The above is really true only in the case of simple poles. For multiple poles there is a more elaborate theory involving all the negative coefficients $\chi_{-i-1}(P)$ in the Laurent series; they are related by a bunch of combinatorial identities of which only the highest one is the trace property.

Local index formula

Theorem 5. *In the situation above the following formulae define a cocycle in the* (b, B)*-bicomplex of* $\mathcal{A}$*. Moreover the cyclic class of this cocycle coincides with the cyclic character of* D*.*

In degree n we have up to a universal constant

$$\sum_{q\geq 0, k_j \geq 0} c_{k,q}\chi_q\big(a_0\nabla^{k_1}(da_1)\cdots\nabla^{k_n}(da_n)|D|^{-(n+2\sum k_j)}\big),$$

where $da = [D, a]$ and $\nabla a = [D^2, a]$.

According to Connes-Moscovici it is an easy calculation using the Getzler calculus to show that this formula defines the $\widehat{\mathcal{A}}$ genus in the ordinary manifold case. Here the terms with $k > 0$ do not contribute.