

# Type III Differential Topology

Nigel Higson and John Roe

Penn State University

higson@math.psu.edu, roe@math.psu.edu

June 28, 2000

## Abstract

This lecture is an exposition of the work of Connes et al on the transverse fundamental class. Basic reference: *Cyclic cohomology and the transverse fundamental class of a foliation*, in 'Geometric methods in operator algebras', Pitman Research Notes volume 123.

# Topology of foliations

A foliation on a manifold  $M$  can be completely described by the subbundle  $F \leq TM$  of vectors that are tangent to the leaves. However, not all subbundles can occur, but only those that are *integrable* in one of the following equivalent senses:

- The sections of  $F$  form a Lie algebra of vector fields;
- The 1-forms annihilating  $F$  form a differential ideal.

Bott (around 1970) asked: Are there topological obstructions to integrability?

He proved the following theorem: If  $Q = TM/F$  is the *normal bundle* to a foliation of codimension  $q$ , then the Pontrjagin ring of  $Q$  vanishes in degrees  $k > 2q$ .

This is carried out by means of Chern-Weil theory applied to a special connection — a *basic connection* — on  $q$ .

## Secondary characteristic classes

Bott vanishing allows the construction of secondary characteristic classes of a foliation. Namely, one can compare the representatives for primary characteristic classes of  $Q$  built from a Riemannian connection, on the one hand, and a basic connection, on the other. Using Bott vanishing one obtains canonical closed forms which implement homotopies between these representatives — secondary characteristic classes.

**Example** Let  $F$  be a foliation of codimension one. Then it is defined by a single 1-form  $\omega$ . Frobenius' theorem gives  $d\omega = \omega \wedge \theta$  for some 1-form  $\theta$  (which may be interpreted as the connection form of a basic connection). Then  $\theta \wedge d\theta$  is a closed form (its exterior derivative  $(d\theta)^2$  is a Chern-Weil representative of  $p_1$  relative to a basic connection, hence vanishes.) Its cohomology class is independent of the choices and is called the *Godbillon-Vey* invariant  $GV \in H^3(M; \mathbb{R})$ .

(Thurston)  $GV$  measures 'helical wobble'. It can vary continuously.

# The transverse fundamental class problem

Let  $M$  be a compact manifold. One can define an ‘integration map’ in  $K$ -theory by

$$x \mapsto \int_M \text{ch } x, \quad K^*(M) \rightarrow \mathbb{R}.$$

This integration map is defined cohomologically. One could also find a family of integration maps reflecting the various characteristic classes of  $TM$ .

Alternatively one could use index theory ( $K$ -homology) to produce an integration map.

Investigate a foliated manifold  $(M, F)$ , or alternatively an action of a group  $\Gamma$  on  $V$ . In either case we would like to produce similar ‘integration maps’ from  $K_*(C_r^*(M, F))$  to  $\mathbb{C}$ .

## About the solution

This problem is solved using cyclic cohomology in the transverse fundamental class paper. Moreover one can reflect the secondary characteristic classes (like  $GV$ ) of the transverse space in the integration map. The continuous variation of  $GV$  shows that these can't arise from  $K$ -homology.

The problem is entirely one of *analysis*.

The natural approach would involve a 'transverse Dirac operator', but this makes no sense without an invariant (transverse) Riemannian structure. As we have seen, the secondary classes (like  $GV$ ) measure the obstruction to the existence of such a structure.

We are in a type III situation.

# What use is the transverse fundamental class?

One needs (a) things to pair with it, and (b) methods to evaluate the result.

One pairs the transverse fundamental class with indices of elliptic operators (this can be systematized by using the Baum-Connes assembly map). And one evaluates the result by an appropriate index theorem.

**Example** Let  $M$  be a manifold provided with a spin foliation. Then there is a transverse fundamental class functional  $\phi: K_*(C^*(M, F)) \rightarrow \mathbb{R}$  such that  $\phi(\text{Index } D_F) = \hat{A}(M)$ , where  $D_F$  is the leafwise Dirac operator. (Of course, the right hand side need not be an integer.)

**Corollary** If  $M$  has nonzero  $\hat{A}$  genus, then it admits no spin foliation whose leaves are of positive scalar curvature.

*Challenge:* Give an elementary proof.

# The cocycle

It is easy to produce a cocycle on the smooth groupoid algebra which represents the transverse fundamental class. In fact, it is just given by an equivariant version of the usual de Rham formula for the fundamental class:

$$(f_0, \dots, f_q) \mapsto \int_V f_0 df_1 \cdots df_q.$$

Here we must take the  $f$ 's to be elements of the groupoid algebra  $C^\infty(V) \rtimes \Gamma$ ; the operations take place in the algebra  $\Omega^*(M) \rtimes \Gamma$  whose elements are formal sums  $\sum \omega_g U_g$ , the  $\omega$  being differential forms: and the algebraic rules are

- $(\sum \omega_g U_g)(\sum \alpha_h U_h) = \sum \omega_g \wedge g^*(\alpha_h) U_{gh};$
- $d(\sum \omega_g U_g) = \sum (d\omega_g) U_g;$
- $\int(\sum \omega_g U_g) = \int \omega_e.$

## The extension problem

This cocycle gives a homomorphism  $K_*(C_c^\infty(\mathcal{G})) \rightarrow \mathbb{C}$ , we need to extend it to a homomorphism  $K_*(C_r^*(\mathcal{G})) \rightarrow \mathbb{R}$ . (Compare lecture 7.)

First attempt is the theory of  $q$ -traces (lecture 4). These are cocycles for which one has an estimate like

$$|\tau(a_1 dx_1 \cdot a_2 dx_2 \cdots a_q dx_q)| \leq C_{x_1, \dots, x_q} \|a_1\| \cdots \|a_q\|.$$

(the left side can be interpreted in terms of the multilinear functional  $\tau$ ).

**Theorem 1.** *A  $q$ -trace on a dense subalgebra of a Banach algebra extends to define a homomorphism from the  $K$ -theory of the whole algebra to  $\mathbb{R}$ .*

This theory solves the problem for  $q = 1$ .

## An awkward example

Let  $\Gamma = \mathbb{Z}$  acting on  $V = \mathbb{T}^2$  via the hyperbolic matrix  $\alpha = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ . (Notice that the crossed product is then the  $C^*$ -algebra of an exponential solvable group  $\mathbb{Z}^2 \rtimes \mathbb{Z}$ .)

Then the transverse cocycle is *not* a 2-trace. This results from the exponential growth of the powers of  $\alpha$  which are introduced when one tries to commute  $x_1$  and  $a_2$  in the expression  $\tau(a_1 dx_1 a_2 dx_2)$ .

Thus one cannot expect that the theory of  $q$ -traces will be adequate by itself.

# Triangular structures

It is easy to deal with the case when  $\Gamma$  acts by isometries (invariant Riemannian structure).

Next simplest case —  $\Gamma$  acts by triangular matrices  $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ , where  $A$  and  $D$  are orthogonal transformations. We refer to a *triangular structure*. (May also assume the  $A$  direction is integrable. . . )

**Theorem 2.** *The transverse fundamental class can be defined in the case that  $\Gamma$  preserves a triangular structure.*

Even in this case the proof is highly non-trivial. It uses the theory of  $q$ -traces, applied not to the  $C^*$ -algebra but to a Banach subalgebra  $B$  defined as the domain of a certain unbounded operator (generalized modular operator); and  $B$  is shown to be smooth because of the existence of the triangular structure.

## Some K-theoretic technology

Recall the Thom isomorphism in  $K$ -theory: if  $W$  is, say, a complex vector bundle over  $B$  then there is an isomorphism  $K^*(B) \cong K^*(W)$ .

**Proof:** The isomorphism is given by taking the product with the ‘Bott element’  $\beta \in K^0(\mathbb{C}^n)$ , and the inverse isomorphism by taking the product with the ‘Dirac operator’  $D \in K_0(\mathbb{C}^n)$ .

In his work on the Novikov Conjecture, Kasparov gave an abstract theory of such ‘Dirac’ and ‘dual Dirac’ elements. One need not restrict attention to *vector* bundles; one can build them for any bundle whose fibers are contractible nonpositively curved spaces on which the structure group acts by orientation-preserving isometries.

## Transfer to the bundle of metrics

Let  $\Gamma$  act on  $V$  as usual, and let  $W$  be the bundle over  $V$  whose fiber at  $x$  is the space of positive definite quadratic forms on  $T_x V$ . The fiber of  $W$  is a nonpositively curved symmetric space  $GL^+(q)/SO(q)$ , and  $\Gamma$  acts on it. Kasparov theory then provides a Thom map

$$K_*(C(V) \rtimes \Gamma) \rightarrow K_*(C_0(W) \rtimes \Gamma).$$

Moreover

**Lemma 1.** *The action of  $\Gamma$  on  $W$  preserves a triangular structure.*

Now we may define the fundamental class in general by first using the Thom map to reduce to the case where a triangular structure is preserved, and then using Theorem 1.

**Remarks:** (1) The construction is inspired by the modular theory of type III von Neumann algebras; (2) There is also an analogy to the ‘transfer to the disc bundle’ used by Farrell-Jones.

# Hypoelliptic theory

There is a way to define the transverse fundamental class as a  $K$ -homology cycle, using the theory of hypoelliptic operators (example: the heat operator). This approach *also* proceeds first by using the transfer to the bundle of metrics to show that we may assume an invariant triangular structure.

Basic idea: as well as the familiar ‘first order signature operator’  $d + d^*$  on a manifold  $M$ , there is also a ‘second order signature operator’  $dd^* - d^*d$  which has the same index.

If  $M$  is a triangularized manifold we may now define a ‘mixed signature operator’ which is equal to the second order operator in the  $A$  directions plus the first order operator in the  $D$  directions. Then the deviations from  $\Gamma$  invariance become lower order terms, and we can define a  $K$ -homology class.

Ref: Hilsum and Skandalis, *Morphismes  $K$ -orientés d’espaces de feuilles et functorialité en théorie de Kasparov*, Ann Sc Ec Norm Sup **20**(1987), 325–390.

# The Godbillon-Vey Class

Our construction of the fundamental class started with a  $\Gamma$ -invariant closed current on  $V$ .

Such a current is a particular example of a cocycle in the groupoid de Rham cohomology of  $\mathcal{G} = V \rtimes \Gamma$ . All such cocycles give cyclic classes for the groupoid algebra, and a particular example is the ‘Jacobian cocycle’ (we assume codimension one)

$$\ell(\gamma) = \log\left(\text{Transverse distortion}\right).$$

This cocycle gives rise to a ‘Godbillon-Vey transverse fundamental class’.

**Theorem 3.** *The GV transverse fundamental class can be reconstructed from the action of the modular group on the standard transverse fundamental class.*

**Corollary 1.** *Suppose  $GV \neq 0$ . Then the von Neumann algebra of the foliation is of type III (in fact one can prove it is of type  $III_0$ ).*